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ABSTRACT

Presented is a summary of research and recommendations concerning questions that are frequently asked by teachers and administrators about the learning and teaching of mathematics. Six areas are analyzed: planning for instruction, instructional procedures, differentiating instruction, methods of instruction, problem solving, and evaluation. The format used was to approach each of the six main headings through a series of questions. A bibliography is included. (MP)

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76 Questions:
A Synthesis Of The Research On Teaching And Learning Mathematics

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INTRODUCTION AND OVERVIEW

This document is a summary of research and recommendations concerning questions that are frequently asked by teachers and administrators about the learning and teaching of mathematics. The intent of the document is to provide a concise summary of major studies having direct implications for planning and teaching mathematics and thus proving to be useful to education.

The document presents a synthesis of selected findings which relate mathematics education to certain areas of concern broadly delineated under the following six headings.

1. Planning for Instruction
2. Instructional Procedures
3. Differentiating Instruction
4. Methods of Instruction
5. Problem Solving
6. Evaluation

One readily notes that any attempt to analyze and associate mathematics education with the set of possible topics spanned by the above limited headings constitutes an awesome, if not impossible, task. In large part, this is due to a recent, massive increase in the attention paid by researchers to the mathematics educational enterprise. In addition to mathematics educators, workers in such fields as psychology, evaluation and measurement, and science education also frequently include mathematical components in their investigations. Consequently, the authors of this document would, at best, sample topics that—in their opinion—had some potential relevance for comtemporary instructional and programmatic developments.

The format used was to approach each of the six main headings through a series of questions. For example, the first question under Planning for Instruction addresses the impact of the mathematics curriculum movement over the two recent decades.

It is the authors' hope that this document will aid the readers to obtain greater focus and understanding of the many issues confronting mathematics education.

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1. PLANNING FOR INSTRUCTION

1.1 What has been the impact of the mathematics curriculum reform movement over the past 20 years?

The revolution in school mathematics during the 1950's and 1960's embraced two important issues: "what mathematics should be taught at what level" and "how mathematics should be taught." The first issue—what mathematics should be taught—has received continual debate during the past 20 years. Since both mathematics and society are in a constant state of flux, it is reasonable to expect that this will be a continuing issue in the years to come. The issue of how mathematics should be taught also has received considerable attention, but it is less well understood and has been somewhat diffused in the content turmoil. Both of these issues are critical to any discussion concerning the efforts to improve school mathematics.

The first new math curriculum projects were designed for college preparatory high school programs. Curriculum projects such as the University of Illinois Committee on School Mathematics (UICSM) and the School Mathematics Study Group (SMSG) producd course materials that incorporated most of the recommendations of the Commission on Mathematics of the College Entrance Examination Board (CEEB) (1959). The widely used SMSG text materials included a treatment of inequalities along with equations and emphasized structure and proof in algebra. Plane and solid geometry were integrated and a section on coordinate geometry was added. Trigonometry was integrated with the second course in algebra and a twelfth grade course in elementary functions was suggested.

The changes in the high school curriculum soon led to efforts to reorganize the junior high and elementary school programs. The programs were designed to be more meaningful and to facilitate an understanding of the mathematics studied. Rote learning was de-emphasized and major ideas of mathematical structures were embedded throughout the materials. Jerome Bruner's *Process of Education* (1960) was influential in providing support for the emphasis on structure and the movement toward treatment of some topics at earlier grade levels.

The overall effects of the curriculum reform are difficult to assess. New text materials, instructural procedures and organization flooded the marketplace after the stimulus from federally funded projects such as SMSG. However, it is unclear what effects these changes had on standard procedures of teaching school mathematics. Unfortunately, the techniques for curriculum evaluations were in their infancy, so a great deal more effort was devoted to developing and implementing various curriculum projects than to conducting well-designed research to study the impact of the projects on children. In general, there is very little conclusive evidence on the advantages and disadvantages of the new programs. Callahan and Glennon (1975) summarized the studies, comparing student performance in new and traditional programs by noting that students in programs emphasizing conceptual notions in school mathematics tend to perform higher on tests composed of conceptual tasks. Likewise, they noted that students in programs emphasizing less conceptual notions demonstrated higher performance on the less conceptual tasks.

The most comprehensive evaluation program was the National Longitudinal Study of Mathematical Abilities (NLSMA) conducted by SMSG. This study collected a vast amount of information over a five-year period, and a large number of psychological variables believed to be related to mathematics achievement were measured. Begle and Wilson (1970) noted that the NLSMA data confirmed that the differences between the SMSG textbooks and the conventional textbooks were largely a contrast of computation level scales and the understanding of mathematical ideas. They reported that the results for the SMSG materials were, in general, favorable. However, not all modern textbooks produced similar results. In fact, some did rather poorly at all levels. These texts tended to be more formal and rigorous than the SMSG textbooks.

Comprehensive studies such as NLSMA have emphasized the multivariate nature of mathematics achievement. The concept of program evaluation has also been clarified by the contribution of Cronbach (1963) and Scriven (1967).

Efforts to assess the impact of the modern mathematics program is not an easy task. The National Advisory Committee on Mathematics Education (NACOME) (1975) tried to estimate the impact of modern mathematics on school curricula by examining data from the National Center for Education Statistics and National Assessment between 1960 and 1972. The results of this investigation indicated that the number of mathematics courses available at the secondary level had greatly increased and the number of students enrolled in Algebra/Trigonometry and advanced level mathematics had also greatly increased between 1960 and 1972. There were approximately 260,000 students enrolled in a calculus or advanced level mathematics courses in 1972, which was four times the 1960 figure. The NACOME analysis also revealed that roughly 500,000 students had studied one of the modern programs in school mathematics during this period.

The NACOME report noted that the changes in enrollment and course offerings in mathematics at secondary levels were paralleled by changes in state and local curriculum guides. However, the reports raised the critical question as to the **extent** which these content and organizational changes were realized in terms of classroom presentations and evaluation in the majority of the mathematics classrooms.

In an attempt to gain some insight into teaching practice, NACOME analyzed the 1972-1973 National Assessment syllabus and tests. Their findings indicated that the National Assessment syllabus contained sets, inequalities, functions, probability, statistics and logic as well as more traditional topics. However, there were only 15 items dealing with these concepts or skills in the 250 test items used with 17-year-old students.

The NACOME report concluded that this hardly suggested a drastic change over the pre-1960 era. Similarly, NACOME found no indications of substantive changes at the junior high level in an analysis of national assessment and commercial standardized tests. The analysis of the elementary level confirmed a slow change.

The National Council of Teachers of Mathematics (1977) surveyed 3,000 second and fifth grade teachers concerning their mathematics curriculum and instructional practices.

The answers to 120 questions were analyzed for each of the 1,220 respondents. The teachers indicated that they preferred a text that places extreme emphasis on skills and drill (14%), greater emphasis on skills than on concepts and principles (27%), equal emphasis on both (48%), greater emphasis on concepts and principles (2%), extreme emphasis on concepts and principles (0%). The teachers were given the list of 10 topics and asked if their textbooks and state or local curriculum guide and tests included these topics.

For example, the teachers reported that little or no treatment was given in their texts to the metric system (65%), graphs and statistics (52%) and probability (62%). They also reported that the system's objectives did not include the metric system (45%), graphs and statistics (40%) and probability (42%). Fewer than five periods of instruction per year were reported by teachers on the following topics: the metric system (61%), graphs and statistics (55%), probability (74%) and relations and functions (48%).

The analysis conducted by NACOME (1975) and the results of the NCTM survey (1977) suggest that the influence of the modern mathematics movement may not be as great as once anticipated.

1.2 What teaching methods and approaches are most effective?

The influence of the curriculum reform movement stimulated interest in effective methods and approaches for teaching mathematics. Long before the revolution in school mathematics, William Brownell (1935) identified three theories of teaching arithmetic: the drill theory, the incidental learning theory, and the meaning theory. Brownell described meaning theory as the conception of arithmetic as a system of understanding ideas, principles, and processes. According to this theory, learning is not just mechanical facility in computation, but rather an intelligent grasp of relations and the ability to handle the situations with comprehension of both the mathematical and the practical significance. Within Brownell's theory, there was no hestiation in recommending drill when it was needed in instruction. But he recommended drill only after understanding had been established as a means to increase proficiency or transfer. During the period of 1935-1960, considerable research was conducted in relation to meaning with respect to arithmetic instruction. Drawson and Ruddell (1955) summarized the findings of studies on meaningful instruction. They concluded that meaningful teaching generally leads to greater retention, greater transfer and an increased ability to solve problems independently. They also suggested that the research implied that teachers should use more materials, spend more class time on development and discussion, and provide short specific practice periods.

Research since this period has supported these findings. Studies by Shipp and Deer (1960), Shuster and Pigge (1965) and Zahn (1966) confirmed the earlier findings that higher achievement in computation, problem solving and mathematical concepts can be obtained when as much as half the class time is spent on developmental activities. This suggests that time spent on practice can be reduced, and computational skills can be maintained with an increase in understanding and problem solving.

1.3 How much instructional guidance should teachers provide?

The discovery-learning controversy centers on the question of how and what type of guidance should be provided for learners. Jerome Bruner (1960) is undoubtedly the person most closely identified with the learning-by-discovery position. Those favoring discovery advocate teaching with minimal guidance and maximal opportunity for exploration. Those favoring guided discovery emphasize the importance of carefully sequenced experiences with maximum guidance.

The effects of discovery teaching were investigated by Worthen (1968) in a study of 538 fifth and sixth grade students. He found that the expository treatment was superior on initial achievement, and the discovery treatment produced superior results in retention and transfer tests. However, a re-analysis of the data (in Worthen and Collins 1971), using classes as experimental units) changed some of the initial conclusions. A similar investigation by Olander and Robinson (1973) with fourth grade students reported significantly higher scores for the expository group on the computation test, while the discovery group scored significantly higher on the retention test. In a summary of the research on discovery-learning, Callahan and Glennon (1975) concluded that the research has not clarified the role of discovery-learning, but it has provided additional insights into the complexity of the teaching-learning situations.

In general, it is difficult to determine the advantages of discovery procedures and the conditions under which these procedures should be used, since the term "discovery" has been used to describe markedly differing procedures.

1.4 What are the implications of the work of Piaget for improving the teaching of school mathematics?

A phenomenon of the past 20 years has been the volumes of research studies generated by the observations and theories of Jean Piaget. (Inhelder and Piaget, 1958; Piaget and Inhelder 1956). It is a complex task to try to describe the implications of Piagetian research studies in this short paper. Piaget's developmental theory of logical processes has focused attention on the importance of action and experiences with appropriate manipulative objects. His theory has also emphasized the importance of language's role in logical development.

Lovell (1972) suggests that Paiget's developmental theory provides the following implications for teaching elementary mathematics: increased small group and individual tasks rather than formal whole-class instruction, use of physical materials and games, social intercourse using verbal language (both between children and between adults and children) and moving from relevant symbolization of mathematical ideas from physical situations to working examples (including drill and practice on paper).

It should be noted that researchers such as Baker and Sullivan (1970), Weaver (1972) and Glennon (1974) have cautioned against an overzealous application of Piagetian theory to education. Beilin (1971) provides a good summary of the limitations in applying Piagetain theory to practices in school mathematics.

1.5 What foundation for learning mathematics do children have upon entering school?

Suydam and Weaver (1975) have gathered data from a number of studies of five-year-olds with no prior schooling. Their summary of findings include the following observations.

1. Many children are able to perform rote counting to 10. Some can count to 20.
2. Although some children can do rote counting by 10s, far fewer can count by twos or fives.
3. Most children understand the meaning of "first," and many know the meaning of ordinals through "fifth."
4. Many children can read the numerals for one to 10, and some can write them.
5. Most children can add and subtract when simple combinations are given verbally with or without concrete materials.
6. Most children know something about coins, time, other measures, simple fractional ideas and geometric shapes.

Although the variability of methods used to assess knowledge in different studies makes it difficult to pinpoint what pre-school children know in general, it seems fairly clear that most children do possess a variety of mathematical concepts when they enter school. Possibly one of the most important results of the attempts to assess this knowledge has been the development of instruments and procedures of assessment. The teacher who wishes to make use of such tests can refer to Rea and Reys (1970) and Schwartz (1969) for sample questions and additional references.

1.6 What factors influence the mathematical knowledge of children entering school?

Using an instrument entitled Comprehensive Mathematics Inventory, Rea and Reys (1970) studied the effects of age, previous education and father's occupation on kindergarten entrants. They made the following observations.

1. Older entrants (69-73 months) scored significantly higher than younger entrants (59-65 months).

2. Children with nursery school experience had a higher level of achievement than those with day care or Head Start experience.

3. The child with a parent who had 16 or more years of formal education tended to score on a higher level than the child whose parents had fewer years of schooling.

4. Children with fathers in professional or highly skilled occupations achieved higher scores than those with fathers who were unskilled or unemployed.

5. Neither sibling relationship nor sex had a significant effect on achievement.

After reviewing a number of studies, Callahan and Glennon (1975) suggested that the older child (within the typical range of entrants) seems to have a higher level of achievement, as measured by standardized tests, than the younger child with the same educational experience. They noted that chronological age may be a more important factor for boys than for girls. Further, a child of average or below average intellectual ability has a better chance of achieving at a satisfactory level throughout the school years the older he/she is within the usual range of ages.

Suydam and Weaver (1975) list IQ and socioeconomic level of parents in studies that they reviewed.

At least one study (Haines, 1961) indicated that entering first graders with kindergarten experience scored higher on achievement than those who did not have kindergarten experience.

In summary, although not all research studies provide consistent results, teachers should be aware that chronological age may be a factor in the achievement of the child entering school. In addition, a child's previous school experience, as well as parental education and occupation, may be factors.

1.7 What is the best predictor of achievement in mathematics at the secondary level?

Over the years, considerable research has been devoted to the problem of predicting achievement in mathematic courses at the secondary level. A variety of studies have assessed the effectiveness of using such factors as mathematics achievement scores, Collins (1967), Hilton and Meyers (1967); aptitude or prognosis test scores, Hanna, *et al* (1969), Kohli (1969); Intelligent Quotients (IQ's), Dirr (1966), Duncan (1961); previous grades in mathematics, Hanna (1967), Barnes and Asher (1962), Anglin (1966) and combinations of these and other factors. In general, the two best predictors of achievement have been previous mathematics achievement and IQ.

1.8 What are some common characteristics of low achievers?

Reluctant learners have been defined in varied terms, including mathematical achievement, grades, reading level, IQ range and other criteria. In general, low achievers are deficient in cognitive functioning (weakness in intellectual skills) and/or affective functioning (poor attitude or self-image). By any definition it is important to realize that reluctant learners are unique individuals with varying strengths and weaknesses. There are no categorical lists of characteristics that describe any individual.

Hoffman (1968) suggested that social and emotional problems often cause difficulty in learning mathematics. These problems are reflected in characteristics such as high rates of absence, low motivation, antisocial behavior, short attention or interest span, inability to see practical uses of mathematics, and lack of goals based on a view of the future.

Schulz (1972) emphasized the importance of realizing that a slow learner is likely to develop a poor self image of himself as a learner and as a person. In terms of cognitive variables, Schulz noted the following characteristics as documented in research literature: improvished language-symbolic system such as limited vocabulary and faulty grammar; inability to abstract symbols; deficient formal speech patterns; and restricted reading and listening comprehension.

Pikaart and Wilson (1970) concluded that operationally diverse classification criteria substantiated the absence of any single method for identifying slow learners. They suggested that students can be best identified on the basis of specific learning difficulties.

1.9 What teaching approaches and materials are most effective with slow learners?

A wide variety of approaches and materials have been used with low-achieving mathematics students. Woodby (1965) found common elements in programs for low-achievers throughout the nation, including mathematics laboratories; use of calculators to find a pattern or an error in computation; a pattern of activities (for security); a change of activities (for attention span); reinforcement of basic concepts; use of manipulative devices; controlled use of games, puzzles and motivational techniques and, where possible, the use of computer-aided instruction units.

Stenzel (1968) found that motivation was greatly improved by using desk calculators and grocery-store-type computing scales in the classroom. Shoemaker (1969) developed calculator materials designed for use in mathematics laboratory. He noted that with proper guidance, learning quickly moves to an atmosphere of problem solving and discovery when the calculator is used to do mathematics.

Scott (1970) found that programmed materials on computational skills, selected on the basis of diagnosed needs, aided under-achievers more than regular classroom instruction did. Bobier (1964) found that low-achieving students often were not motivated sufficiently to work independently in programmed books.

However, Sherer (1967) reported that students using materials with instructional aids such as drawings, counters, number lines and charts, along with tutors, gained significantly in arithmetic achievement.

School systems both large and small have produced a variety of curriculum materials for low-achievers. Hoffman (1968) described a program for low-achievers in the Jefferson County, Colorado, public schools which included the following components: multiple activities (completed and evaluated in class); electric printing calculators; problems presented on letterhead from local businesses; flow charting; mathematics laboratory experiments; multi-sensory aids, including scales, slide rule models, audiovisual equipment; and laboratory involvement projects, including puzzles and kits. Hoffman reported that the evaluation of the program revealed aroused interest and skill development of students who were formerly disinterested and unmotivated.

Kneitz and Creswell (1969) reported an average gain of seven months in two months' instruction for 66 high school dropouts in the Houston area. They also reported noticeable changes in attitudes, poise and dress. Their program included individualized materials, programmed booklets, computational skill kits, crossnumber puzzles and frequent changes of pace, such as filmstrips, math games and real-life problems.

Brain (1965) in the USOE Report on slow learners emphasized the vital role of administrators and administrative policies in implementing programs for low achievers. He enumerated the following guidelines.

1. Provide staff and community orientation programs.

2. Provide adequate course offerings at each grade level.

3. Use appropriate grouping procedures.

4. Provide assistance to teachers (paraprofessional support, materials, released time, tutors, etc.).

5. Involve parents and community support personnel.

6. Give special consideration to selection of teachers.

7. Provide inservice programs.

In general, an analysis of the characteristics of low-achievers and the materials and approaches designed for them suggests the importance of developing success-oriented programs with provision for individualized work that employs an extensive variety of learning aids. In addition, careful consideration should be given to such factors as (1) causes of learning difficulties, (2) developmental stage of the child, (3) methods or modes that are best for students in relation to the difficulties encountered, (4) factors from home, school and community that enhance or hinder learning and (6) classroom administrative techniques.

Travers, LeDuc, and Runion (1971), in a review of the literature on teaching resources for low-achieving mathematics classes, summarized the approaches that have been found to be particularly effective. Their summary included (page 28)

1. Employ repetition through spacing.

2. Take small segments.

3. Use language which the class can understand.

4. Dramatize the material.

5. Individualize the problem assignments.

6. Pay attention to reading the problem.

7. Provide variety within the class period.

8. Use concrete approaches.

9. Provide activities.

10. Hold frequent reviews.

11. Use praise freely.

12. Display good student work.

13. Build assignments that lead to success.

14. Use diagnostic testing before and after teaching.

15. Measure the student against himself/herself.

16. Correlate mathematics with other subjects.

17. Establish consistent classroom management policies.

18. Use audiovisual techniques when possible.

19. Try supervised study rather than homework.

20. Grade work the day it is turned in.

21. Do not insist on verbalization.

22. Allow the use of calculators or tables.

1.10 What is meant by the term attitude and how is it measured?

The following definition of "attitude" was developed at the Educational Testing Service: ". . . . an attitude is an implicit cue- and drive-producing response to socially salient characteristics and . . . . it possesses evaluative properties" (Anderson, Ball, Murphy, 1975, 32). An attitude is within the individual (**implicit**). An attitude will cause the individual to behave selectively (**cue- and drive-producing**). An attitude can be elicited through the appropriate stimuli (**response**). An attitude may be assessed with respect to school-related activities (**socially salient**). An attitude contains either a positive or negative element (**evaluative**).

Since 1970, there has been a plethora of studies related to attitudes toward mathematics. The two most widely used measures are the Thurstone-type instrument (Dutton, 1951) and the Likert-type (Aiken and Dreger, 1961). When attitude scores are used to predict achievement, a low but significant positive correlation is usually found (Neale, 1969). Aiken (1976) reports that this result has been found at the elementary, secondary and college (undergraduate and postgraduate) levels, as well as with students in other countries and with minority students within the United States.

While there appears to be a direct relation between attitude and both actual and aspired marks (Spickerman, 1970), attitude and grade level are inversely related (Callahan, 1971; Evans, 1972; Jacobs, 1974). The correlation between attitude and achievement is generally higher for girls than for boys (Behr, 1973); hence, girls' mathematics grades are more readily predicted from their attitude scores than are boys'.

1.11 What is the effect of teacher attitudes on student attitudes?

Lewis R. Aiken (1970) claimed that of all the factors which affect student attitudes toward mathematics, teacher attitudes are of particular importance. In a 1976 review of the literature related to attitudes, he said, "The belief that teachers' attitudes affect students' attitudes toward mathematics has not been as easy to confirm as might be supposed" (Aiken, 1976, 299). He also reported that several studies have found no statistically significant relationships between either attitudes of students or attitude changes of students at the elementary school level.

Phillips (1973) found that the type of teacher attitude a student has encountered during the past two or three years and especially the most recent teacher's attitude, is significantly related to the student's attitude at a given time.

As might be expected, the effect of teacher behavior on student attitude varies from teacher to teacher. Such behaviors as failing to announce examinations or writing comments on student papers may result in unanticipated student attitudes, and these attitudes may vary from student to student.

1.12 What is the role of a mathematics teacher in enhancing student motivation?

Motivational research is the study of what conscious or subconscious influences induce people to choose or reject a course of action. Students who are motivated in mathematics exhibit various observable behaviors such as (1) reading mathematics books for pleasure, (2) working on projects or problems not specifically assigned, (3) exhibiting enthusiasm about mathematics, (4) persistence toward a task, (5) thoroughness in study and research, (6) concentration on work and (7) attention and participation in class (Skemp, 1973; Fowler, 1976).

In his book *Beyond the Information Given* (1973), Bruner theorizes that learning is what students want; therefore, each student has an internal drive to achieve expertise in some particular area of endeavor. Motivation lies in the student's enjoyment of the transformation of surprise and complexity to predictability and simplicity and arises from putting knowledge to work in exploring a topic that is important to the student now.

According to Davis (1973) a student is motivated by making his/her own discovery and verifying that what was discovered actually checks out. He feels that motivation arises in a competitive setting.

Dienes (1973) contends that a creative learning environment is self-motivating. If the teacher creates a relaxed, free atmosphere in the classroom, the students will be motivated to learn.

According to Skemp (1973) short-term motivations such as teacher approval and fear of displeasure are effective in the early school years. For those students who already enjoy mathematics, internal motivations are sufficient. For those who do not want to learn mathematics, extrinsic motivations are necessary.

Often students with apparently poor attitudes toward mathematics do not really dislike mathematics; rather, they hate drudgery, frustration and boredom. With mental ability held constant, differences in achievement are frequently the result of differences in motivation. Recommended ways of motivating students are as follows.

1. Teach with enthusiasm. Enthusiasm is contagious. When backed by enlightened and empathetic competence, enthusiasm is perhaps the only real guarantee of effectively maintaining student interest.

2. Use contemporary materials. The use of newspaper items, magazine articles and school activities heightens student involvement in the learning process.

Motivation may also be enhanced by providing a creative classroom environment that is flexible and relaxed; where the student is challenged to reach his intellectual potential. The ultimate goal is to cause the student's primary driving factor to be his enjoyment, interest and curiosity of mathematics.

To summarize recent research (Bruner, 1973; Davis, 1973; Skemp, 1973; Fowler, 1976), the teacher should strive to fulfill several roles.

The teacher must be aware of the student's needs and interests—a good **listener**. The teacher needs to know the potential ability of each student—a good **evaluator**. The teacher must know mathematics well enough to relate it to problems and situations that are real to the student—

a good **scholar**. The teacher should be able to use various teaching strategies to enhance student interest—a good **inventor**. The teacher needs to be able to create an atmosphere conducive to learning—a good **innovator**. The teacher should try to influence the student's motives so that his/her prime motivational drive is that of learning mathematics for enjoyment, pleasure, interest and usefulness—a good **problem solver**. Although no teacher is likely to fulfill all of the above, each teacher should strive to motivate students with these roles in mind.

1.13 What is the relationship between creativity and ability in mathematics?

There is no universal agreement as to the relationship between intelligence and creativity. Although most researchers agree that a certain minimum level of intelligence is necessary for creativity, a particular level of intelligence does not appear to be sufficient for predicting creative endeavors.

In general, correlations of certain tests of creativity (such as Torrance's creativity test) and intelligence tend to be low. For example, Lanier (1967) and McGannon (1972) found low correlations of general creativity and intelligence tests. In contrast, moderate to significant relationships exist when certain creativity tests are correlated with intelligence tests or mathematical achievement tests. Creativity tests prepared by Guilford, *et. al.*, were found by Borgen (1971) to be significantly related to arithmetic achievement. Evans (1965) and Banghart and Spraker (1963) found significant correlations between their achievement tess in mathematics and creativity.

There are questions about the reliability and validity of many measures of creativity. Reliability of most measures of creativity tend to be moderate, usually in the .80's, and correlations among different creativity tests are frequently low, which leaves many questions about what these instruments measure and contribute to the mixed results of research.

The debate continues as to the relationship between creativity, intelligence and achievement. Some researchers such as Thorndike (1963) found evidence to support the existence of a separate creativity factor. Others, such as Burt (1962), Richards and Bolton (1971) and Yamamoto (1965), and Lucito (1967), viewed creativity as a part of eneral intelligence.

Since the question of the relationship between creativity and intelligence is not settled, further questions related to the relationship between creative ability and problem solving ability are crucial.

1.14 What is known concerning sex differences with regard to attitudes, parents and self-concepts as related to mathematics?

In October 1976, John Ernest of the University of California at Santa Barbara reported the results of an undergraduate research seminar in which the students attempted an impartial examination of sex differences in mathematics. One aspect of their research centered on student attitudes with a sample of 1,324 students in grades two through 12.

When students were asked to rank four subjects (mathematics, English, science and social studies) from most-liked to least-liked, boys tended to prefer science while girls tended to like English. Social studies had a higher rating from boys than from girls. Mathematics was the only subject which showed no difference in preference by male or female. Aiken (1972) also reported this same outcome with eighth graders. However, when students have an option of taking mathematics course (in secondary school and college), more boys than girls tend to enroll in mathematics courses. Lucy Sells (1973) considered a possible explanation for this to be an awareness of the need of mathematics in future endeavors rather than a change in liking. If this conjecture is true, then an upgrading of the high school counseling programs, where an attempt is made to make female students more aware of career opportunities, might have some effect on the number of mathematics courses elected by females.

Ernest (1976) also reported on which parent gave help in the various subjects. There were no statistical differences between boys and girls. For both boys and girls, mothers helped more than fathers until the upper grades (beginning in grade six) when the fathers became the mathematics "authority." Nancy Krienberg (1977) reported on a series of conferences in the San Francisco area to attract young women to study mathematics and science. Included in the conferences were sessions for parents and teachers. The purpose of these sessions was to aid the adults in reducing sex stereotyping and to dissiminate practical information on assisting young women in pursuing mathematics and science interests.

The third question examined by Ernest's seminar (1976) was which sex most students in grades two to 12 considered most proficient in various school subjects. In the lower grades, the boys thought boys do better while the girls thought that girls do better in all subjects. By secondary school 32 percent of the boys and girls together thought that boys do better in mathematics, 16 percent thought girls do better and 52 percent said they thought there was no difference. In general, females are more apt to underestimate their intellectual activities (Maccoby and Jacklin, 1973) as well as their own problem solving activities (Kagan, 1964). Sanford Dornbusch (1974), a sociologist, found that when asking students the reason for getting a poor grade in school, most students gave lack of effort except in mathematics. Twenty-six percent of the female students gave lack of ability (as opposed to 15 percent of the male students) as the basis for a poor grade in mathematics.

1.15 What is known concerning sex-difference in relation to mathematics?

The notion that boys do better than girls in mathematics has currently received some attention. In 1974, Fennema found that there were no significant differences between boys and girls before entering elementary school or in the early years of elementary school. Nor did significant differences always occur at the upper level elementary or high school years. Howwever, when they did apper they "were more apt to be in the boys' favor when higher-level cognitive tasks were being measured and in the girls' favor when lower-level cognitive tasks were being measured" (Fennema, 1974, 137). Maccoby and Jacklin (1974) made a stronger claim, stating that "boys excel in mathematical ability" (Maccoby and Jacklin, 1974, 352). They failed, however, to clarify between ability and achievement. Additionally, they point out that boys excel in visual-spatial ability and that this difference does not appear until adolescence. An interesting note is that visual-spatial difference based on sex does not occur in all cultures. Kabanova-Meller (1970) reported that this difference does not appear between Russian boys and girls in grades four, five and six. Berry (1966) and Kleinfeld (1973) found no spatial ability differences between Eskimo males and females.

With regard to visual-spatial abilities, it is important to consider that many mathematicians believe that all mathematical thought involves geometrical ideas. Bronowski (1947) said that the total discipline of mathematics could be defined as the language for describing those aspects of the world which can be stated in terms of configurations. Meserve (1973) said that those who used mathematics used modes of thought of geometry and that "even the most abstract geometrical thinking must retain some link, however attenuated, with spatial intuition" (Merserve, 1973, 249). Fennema and Sherman (1977) point out that the development of visual spatial skills may be closely related to sexual stereotypes and therefore may be largely a result of the culture.

If one agrees with the above, then merit should be given to the recommendation of Ernest (1976) that these considerations have implications for school curricula.

Tittle (1973) has shown that many achievement tests are sexually biased. Certainly if a mathematics test contains items that require spatial ability, the possibility exists that girls will do less well than boys. Fennema (1975) suggests that if a test were constructed that had little or no spatial content, perhaps no sex difference would be found. "On the other hand, spatial visualization may be such an integral part of higher mathematical thinking that eliminating spatial aspects of mathematics tests too narrowly restricts the area of mathematical thinking" (Fennema, 1975, 40).

Clearly this suggests further investigation of spatial visualization (a) in the elementary school curriculum (particularly for girls) and (b) in testing.

1.16 To what extent does sexual stereotyping occur with regard to mathematical performance?

Sex-related differences in mathematics have been used as a variable in many studies. Reviews of the literature prior to 1974 indicated that although there did not appear to be a sex-related difference in young children, by the upper elementary level a difference became evident. National Assessment of Educational Progress (NAEP), (Mullis, 1975), reported sex-related differences in mathematics performance. "In the mathematics assessment, the advantage displayed by males, particularly at the older ages, can only be described as overwhelming" (Mullis, 1975, 7).

1.17 How is student anxiety level in mathematics classrooms related to other variables?

Anxiety has long been a central concern in the area of learning and academic performance. However, Ohlson and Mein (1977) recently noted that "Although these studies point out the relationship between anxiety and numerous areas of human behavior, there is a lack of research relating anxiety level to various academic disciplines, particularly mathematics" (Ohlson and Mein, 1977, 48). The Ohlson and Mein study attempted to determine whether a difference existed in the degree of anxiety possessed by undergraduate mathematics majors as compared to undergraduate non-mathematics majors. For these two groups, a secondary purpose of their study was to investigate the extent to which difference in sex influences anxiety level. Ohlson and Mein concluded that mathematics majors seem able to cope with classroom situations even if anxiety is present. For non-mathematics majors in a three-quarter sequence the teacher did appear to influence the students' attitudes toward the mathematics class. This supports Aiken (1970) who stated, "Of all the factors affecting the student attitudes toward mathematics, teacher attitudes are viewed as being of particular importance." (Aiken, 1970, 592).

It appears that stressful conditions are necessary to produce significant anxiety levels in students. Szetela (1973), in a review of studies using programmed materials, reported that there appears to be little association between anxiety toward text materials and learning mathematics. The programmed materials provided immediate feedback without teacher threat, so a stressful condition did not really exist. Szetela (1973) points out concerning his own study that anxiety measuring instruments may not adequately measure what was purported as measured, and he suggests that this be kept in mind when one is considering studies in this area.

Both Hodges and Felling (1970) and Ohlson and Mein (1977) found that neither sex difference nor achievement variables contributed toward predicting anxiety levels. One explanation for finding that student achievement is apparently independent of anxiety level was suggested by Jackson.

> *"Suppose . . . that a small number of students dislike school intensely and an equally small number are correspondingly positive in their opinion, but . . . most students have either mixed or very neutral feelings about their classroom experience. Perhaps for attitudes*

> *to interact with achievement they have to be extreme, and extreme attitudes, either positive or negative, may be rarer than is commonly though." (Jackson, 1968, 81)*

An important question which one can pose after reviewing the literature on anxiety studies is the following: Do these studies imply that everyday classroom routines result in measurable anxiety scores? Perhaps, as Jackson indicates,

> *". . . certain kinds of extreme feelings may not appear too frequently in the classroom . . . If school is inevitable, better relax and accept it. One reason why . . . attitudes toward school tend toward neutrality is that school becomes 'old hat' for most students . . . The excitement of school, its sharp disappointments as well as its joys, is contained in colorful interludes that interrupt, rather than characterize, the normal flow of events." (Jackson, 1968, 61)*

In short, it may be, as Ohlson and Mein suggest, that negative remarks made by students about mathematics are in fact expressions of dislike, fear or anxiety toward the classroom atmosphere rather than the subject matter.

If one accepts the fact that increased mathematics anxiety should result in greater "math avoidance," then the following study is germane. With regard to sex difference, Sells (1973) pointed out that there is evidence of greater anxiety among females than males. Of the entering class at Berkeley in 1973, 57 percent of the males brought with them four years of high school mathematics, but only eight percent of the entering females had the same preparation. Thus, 92 percent of the women in the first-year class were not able to take any calculus or intermediate level statistics course. Moreover, all but five of the 20 majors at Berkeley required either calculus or statistics. Women, then, were crowding into the remaining five fields (the humanities, music, social work, elementary education and guidance and counseling), primarily because of avoiding mathematics at the secondary school level. More discussion of this topic will be found in other sections of this paper which address sexual stereotyping and sex differences.

2. INSTRUCTIONAL PROCEDURES

2.1 What organizational patterns are most effective?

Research efforts to isolate and measure the effectiveness of organizational patterns such as departmentalization, non-graded team teaching, multi-graded and self-contained classrooms are extremely difficult. A variety of factors interact with these patterns, and achievement is affected by many factors, not the least of which is the teacher. Hart (1962) reported a study of 50 beginning fourth grade pupils who had spent three years in a graded primary program and 50 matched pairs of pupils who had spent three years in a non-graded primary program. He found that the non-graded group achieved significantly higher than the graded group. Steere (1967) reported that tenth grade students in graded schools had gained significantly more in mathematical reasoning than had students in non-graded schools.

Wolff (1968) compared third-year students in individualized graded and non-graded classrooms. He found no significant differences in arithmetic achievement. Paige (1966) reported no significant difference in mathematics achievement, retention or re-learning ability between students taught by team teaching and those taught by a single teacher. Price, Prescott and Hopkins (1967) studied the effect of special subject teachers at the fifth grade level in a carefully designed experiment. There were no significant differences in achievement between students in the departmentalized classrooms and those in the self-contained classrooms. Morrison (1967) found that students in self-contained classrooms scored higher on reasoning and computation tests than did those in departmentalized classes. Both Willcutt (1969) and Backman (1969) reported no significant achievement differences between seventh-graders in self-contained classes and those in ability-grouped classes.

More recently, there has been considerable interest in assessing the effectiveness of "open education." Earnshaw (1973) investigated the effects of an informal learning environment on student attitudes and motivation. The open education program measurably influenced students in terms of resourcefulness, creativity, initative, self-reliance and enjoyment of school. However, the students in the open education program did not score as well on standardized mathematics achievement tests as students in the regular classroom did.

In general, the studies on class organization are action-research, which implies less firmly controlled variables; thus, the findings are equivocal. Suydam (1972), in a summary of research on classroom organizational patterns, notes that the one generalization that can be made is that depending on the variables involved any organizational pattern can be effective and can produce significant achievement. Suydam (1972, p. 5) also concludes that the most important factor is the teacher. "If the teachers are committed to a particular pattern, they can make it work. Conversely, some teachers can make any pattern work," he said.

2.2 What is the most effective use of students' and teachers' time?

The classroom teacher normally makes decisions about how much class time should be devoted to certain activities. In general, research suggests that more time should be spent on developmental activities and discussion than on practice, and that short and specific practice periods should be provided. Ship and Deer (1960) compared groups in which 75 percent, 60 percent, 48 percent or 25 percent of class time was spent on group developmental work while the remainder was spent on individual practice. Higher achievement was obtained on measures of computation, problem solving and concepts when more than half of the time was spent on developmental activities. This experiment was replicated by Shuster and Pigge (1965), and the findings were confirmed.

Zahn (1966) compared four different treatments, ranging from one which favored developmental over practice activities to one which favored twice as many practice activities as developmental ones. The results supported the conclusion that students learn arithmetic skills better by spending less time on practice and more time on developmental activities. This conclusion was valid for all three ability levels considered.

Hansen (1963) tested the effect of lengthening class time from 55 to 110 minutes and of meeting on alternate days. Extended class discussion, a mathematics laboratory, library reading, class reports and more instructional aids were included in the expanded class time. The results indicated that achievement and attitudes of students in the longer classes were not significantly different from those of students in the daily 55-minute classes. The results **tended** to favor the longer class time group.

2.3 How should the sequence of content be determined?

During the curriculum reform movement, considerable attention was given to the reorganization and sequence of content. Research reflects various trials. Gagne' (1968) suggested that any human learning task may be analyzed in terms of component tasks. He suggests that it is necessary to (1) identify the component tasks of the final performance; (2) ensure achievement of each of the tasks and (3) arrange the total learning situation in a sequence which insures optimal mediational effects from one component to another. Gagne' and Bassler (1963) found that sixth grade students learned a concept when it was developed according to a hierarchy of subordinate knowledge. Even though the students did not retain all the subordinate knowledge, they did achieve well on the final task. Buchanan (1972) studied the effect of prior experience with subordinate tasks in relation to mastery of the final task. He found that the amount of prior experience with the introductory task had a significant effect on mastery of the final task. Phillips (1972) developed and evaluated a learning hierarchy for adding fractions with like denominators. He experimented with seven hierarchal orderings and 11 subtasks. He found that the sequence seemed to have little effect on immediate learning; however, long term retention was found more sensitive to sequence.

Suppes (1967) has developed sets of mathematical models described by variables important in the presentation and sequencing of content. Also, Scandura and Wells (1967) have developed approaches to sequencing variables. Heimer (1969) provides a good review of approaches used in sequencing content.

2.4 What techniques are effective in motivating learning in mathematics?

The subject of motivation has received considerable debate. In general, evidence suggests a common-sense hypothesis that short-term motivation, such as desire to please a teacher, is likely to be effective; however, this is extrinsic to mathematics instruction. Skemp (1971) suggested that the need for mental activity or the enjoyment of such activity can become an intrinsic motivation in mathematics instruction. Thus, the ability to increase interest and achievement in mathematics provides motivation. In this sense, materials, games and problem situations that successfully encourage interest can be considered motivational.

Suydam and Weaver (1975) reported that the effect of teacher enthusiasm is a critical factor, as well as what the teacher says and how it is said. They note that praise has been found to be a highly effective technique for providing motivation.

2.5 What techniques are effective in producing transfer and retention?

Transfer is the ability to apply something learned from one experience to another. Transfer appears to be facilitated by instruction in generalizing or by teaching students to see patterns and apply procedures to new situations. Research evidence suggests that meaningful instruction and discovery-oriented instruction facilitates transfer. A study by Kolb (1967) demonstrated that children transferred mathematics instruction when the transfer was carefully planned. Most studies support the common-sense conclusion that transfer is facilitated when teachers plan and teach for transfer.

Another important aspect of learning is retention. In general, research suggests that when something is meaningful to students and understood by them, they are more likely to remember it. A study by Burns (1960) emphasized the importance of intensive and specific review periods for facilitating retention. Schuster and Pigge (1965) reported that retention was better when at least half the class time was devoted to meaningful or developmental activities.

In general, the older a child and the higher his/her ability level, the better the transfer and retention. However, Klausmeier and Cheek (1962) reported that children of various ability levels were able to transfer problem-solving skills and that retention was good when they were given problems at their individual ability level.

2.6 How effective is teaching mental computation and estimation?

Many mathematics educators are concerned that in the mathematics curriculum, far too much emphasis is placed on processes and problems that lead to exact answers. Kramer (1970) noted that approximately 75 percent of adult non-occupational uses of mathematics involves mental calculations. However, Suydam and Weaver (1975) reported that relatively little recent research has been directed toward this increasingly important ability. Payne (1965) reported that an experimental group of fifth grade students whose arithmetic instruction was supplemented by mental computation materials made statistically significant gains over control groups. Pupils in the experimental group worked more problems correctly, both within a 25-minute time period and when no time limit was imposed.

Austin (1970) reported that eighth grade students who spend one period a week on mental computation scored significantly higher on standardized tests than students not receiving this instruction. Grumbling (1971) found that fourth grade students who were instructed in mental computation were better able to solve problems mentally than students not receiving the instructions. The students receiving instruction in mental computation also showed a significant increase in arithmetic achievement. Rea and French (1972) reported that the majority of sixth grade students receiving instruction involving either enrichment or mental computation activities demonstrated a dramatic gain in arithmetic achievement. Scholl (1973) investigated the effects of a variety of modes of presenting short and frequent periods of oral practice. He found that the fifth grade students exposed to this treatment increased in the ability to compute mentally and showed a gain in attitude. He found no significant differences among the groups using televised lessons, lessons on audio-tape, or programmed materials.

A skill that is closely related to mental computation is that of estimation. The *NIE Conference on Basic Mathematics Skills and Learning* (1975) emphasized the importance of estimation as a basic skill. Johnston (1976) noted that a careful review of mathematics text materials revealed that very little or no attention is given to the development of estimation or approximation. The *Cape Ann Conference on Junior High School Mathematics Report* (1973) suggested that esimation is one of the most neglected aspects of the mathematics curriculum in grades six through nine. The report emphasized that the process of estimation is important for the following reasons.

1. Estimation is a useful vehicle for mental calculation and can help prevent gross errors in computations.

2. Estimation is helpful in obtaining a rough approximation which helps in selecting operations and judging the reasonableness of results.

3. Estimation insures that numbers from measurement and counting will be used more meaningfully.

4. Estimation is helpful in developing a better understanding of computational algorithms, particularly division.

5. Estimation is useful in problem solving and everyday life.

Ashlock (1976) noted that estimation is a very complex task and is a skill that requires instruction and practice. He suggested that the following are among the skills necessary for estimation.

1. Round a whole number to the nearest ten, hundred, etc.
2. Multiply by powers of ten.
3. Add, subtract, multiply and divide two numbers, each of which is a multiple of ten.

Johnston (1976) indicated that the ability to make rough approximations becomes critical as students begin to use calculators more in the mathematics curriculum. He noted that if the wrong key on a calculator is inadvertently pressed, the result will obviously be wrong. Being able to estimate the answer decreases the probability that a student will accept an unreasonable result.

Thus, it seems that greater attention should be devoted to developing the ability to perform mental computation and estimation in the mathematics classroom.

2.7 What types of homework assignments are most effective?

Results of studies on the important question of how to use homework have not been consistent. Studies by Maertens and Johnston (1972) and Doane (1973) reported an achievement gain when homework was used. However, the relative effectiveness of various types of homework assignments is unclear. Brinke (1967) studied seventh and eighth grade classes and found that supervised study is more beneficial than homework. Peterson (1970) reported that both a group receiving exploratory homework assigned three days prior to the teaching of a topic and a group receiving mathematical puzzles unrelated to the topics being taught, achieved better than a group receiving no homework. Also, students completing 50 percent or more of the homework assignments achieved more than students who did 50 percent or more of the puzzles. Small, Holton and Davis (1967) reported no significant differences in achievement between groups whose homework was spot-checked and groups whose homework was carefully graded each day. Brown (1966) reported no signficant differences in test scores or homework grades in groups that had received only grades or only conferences on homework in elementary algebra.

Austin (1974) found that comments on homework papers seemed to improve student achievement in certain mathematics classes (geometry and general mathematics). However, comments did not appear to have any effect on achievement in other mathematics classes (algebra and grade four).

Studies comparing homework and no-homework have been few. Goldstein (1960) reviewed research studies from 1900 to 1959 and found only 17 comparing homework with no-homework. Goldstein reported that the evidence tended to support homework as an important factor in increasing student achievement. Austin (1974) reviewed the research from 1900 to 1974 and found only 13 studies comparing homework with no-homework in mathematics. Austin concluded that results tend to support the belief that regularly assigned homework is an important factor in improving student achievement in mathematics.

2 8 What are the three major factors that contribute to the reading of mathematics?

The three major factors contributing to the reading of mathematics are (1) vocabulary (2) experiences in verbalizing mathematics, and (3) the ability to note similarities and differences between mathematics and reading.

Mathematics has long been considered a specialized language—"a language in which human knowledge of the physical world has been recorded" (Aiken, 1977, 251). The letter, word and syntactical redundancies of mathematics are quite different from those of natural languages. "A knowledge of vocabulary is essential in fields like mathematics, where many special terms are used" (Olander, 1971, 361). Warncke and Callaway (1973) reported that research relating reading and mathematics indicates that poor readers have poor problem-solving ability and recommended that teachers give special instruction in vocabulary development in mathematics. The number of new mathematical and scientific words encountered in the elementary school is often overwhelming (Wilmon, 1971). Technical vocabularies and their specialized meanings must not be left to incidental learning (Taschow, 1974).

Knight and Hargis (1977) recommended that the teacher provide optimum conditions and experiences for the development of the language of mathematics through a variety of activities. Earp (1970) recommended that special teaching activities be devised and tested with specific procedures that deal with reading and mathematics. Aiken (1977) provided some suggestions of activities such as "arranging the order of topics, posing problems, asking questions, encouraging discussion and providing opportunities for observation and exploration in laboratory-type situations" (Aiken, 1977, 254).

Collier and Redmond (1974) recommended that teachers actually spend time explaining and illustrating differences between mathematics and materials in other study areas. Hater, Kane and Byrne (1974) have specific suggestions for the classroom teacher on dealing with problems such as eye-movement patterns that differ in mathematics (e.g., mathematics is often read from right-to-left as opposed to the usual left-to-right eye movement required in narrative reading).

2.9 What does research on the vocabulary of textbooks indicate?

In general, studies dealing with the vocabulary of elementary school mathematics textbooks have emphasized the large number of new vocabulary words found at each grade level and the rate at which the new words are introduced.

Hunt (in Buswell and John, 1931) analyzed six third grade books and found a total of 2,993 different words in the collection. Only 350 of the words appeared in all six books. Of 306 technical arithmetic terms, Hunt found only 34 that were in all six books examined.

Repp (1960) studied five third grade textbooks and reported a total of 3,329 different words in the five. Only 698 words were in all the books. She found the average number of new words per page to range from 3.98 to 6.98 with a low of no new words on some pages to a high of 69 different new words on one page.

In a report on the readability of elementary mathematics materials, Heddons and Smith (1964) noted that the average reading levels of the materials considered were generally much higher than their designated grade level, especially at the first grade level.

Smith (1969) used a reading formula to analyze 11 seventh and eighth grade mathematics programs. Great variability of reading levels was found within each series. The reading materials, which had vocabulary ranging from fourth grade to college level, had an uneven distribution of easy and difficult materials throughout the books rather than a progression from easy to more difficult.

Comparing the vocabularies of mathematics and reading textbooks, Stevenson (1971) noted 396 mathematical terms that appeared in the third grade books investigated. While 161 of the words were common to the mathematics books, only 51 were in both the reading and the mathematics books. Reed's study (1965) produced similar results. Of 217 different technical words found in two basic arithmetic series for grades one to three, only nine were also introduced in two basic reading series for the same grade levels.

Various studies on the vocabulary used in elementary school textbooks indicate that teachers should consciously teach the vocabulary employed if they intend for children to read the materials with understanding.

Along a different line, Knight (1971) used sub-culturally appropriate language to teach a unit on nonmetric geometry. The pupils taught by this method performed more successfully than those taught the unit by means of standard English. Teachers should consider the implications of this study when choosing the words to use in explaining mathematical ideas.

2.10 What is the role of manipulative aids and other types of models?

Materials are sometimes classified as manipulative (concrete), pictoral (semiconcrete) and symbolic (abstract). Much attention has been focused on identifying the role of each type. The studies described below represent the body of research dealing with both the effectiveness of manipulative materials and the relative effectiveness of manipulative materials as compared to other types of materials.

Harshman, Wells and Payne (1962) studied for one year first grade children who were taught various content by means of either (1) a set of inexpensive commercial materials, (2) a set of expensive commercial materials or (3) materials supplied by the teacher. Inservice training was provided for teachers using the sets of commercial materials. All significant differences noted favored the third group, which used materials supplied by the teacher. The researchers concluded that expensive materials are not necessary for an effective program. Their results also led them to suggest that different materials possibly should be used with children of different ability levels.

Lucas (1967) investigated the effects of using attribute blocks, varying in shape, size and color, with first grade children. His results indicated that children using the blocks were significantly better able to conserve number and to perceive addition and subtraction relations than were children not taught with the blocks. Children exposed to a more traditional approach scored higher in computation and verbal problem solving.

Working with first graders, Weber (1970) compared the use of pencil and paper follow-up activities with the use of manipulative materials for follow-up activities. On a standardized test no significant differences were noted; however, there was a definite trend in favor of groups using the manipulative materials. On an oral test of understanding the manipulative activities group scored significantly higher than the other group.

Ekman (1967) presented addition and subtraction concepts to children in grade three by (1) having children manipulate cardboard disks prior to learning the algorithms, (2) presenting the ideas through pictures, (3) developing the ideas in algorithm form from the beginning. The group that manipulated objects performed higher on understanding and transfer scales. Although the first two groups performed at a higher level on a skills scale, there was no significant difference at the end of the retention period. Fennema (1972b) studied the relative effectiveness of a meaningful manipulative model and a meaningful symbolic model in teaching a concept to second graders. She measured learning by (1) tests of recall, (2) two symbolic transfer tests and (3) a concrete transfer test. No significant differences were found on any of the tests. Materials chosen appeared to be no more important than the fact that both models presented the mathematical concept in a meaningful way, relating it to prior knowledge of the children.

Using second graders as subjects, Knaupp (1971) studied two instructional methods and two concrete models for developing addition and subtraction algorithms along with the concepts of base and place value. His results indicated that either teacher-demonstration or student-manipulative approaches with sticks or blocks led to significant gains in learning.

Probably more research has focused on the use of the Cuisenaire materials than on any other specific set of materials. Of seven studies examined by Fennema (1972a), three studies showed significant differences in favor of the Cuisenaire materials on the total range of first grade mathematics work; one study showed neither the Cuisenaire nor the conventional treatment better on the total range of second grade work; two studies showed neither treatment better for multiplication and division at the third grade level and one study showed the traditional method significantly better for computation and reasoning at the third grade. The relative effectiveness of the Cuisenaire materials seems to be linked to the grade level of the children. Other factors in the success may be prior background, length of time and the specific topic developed.

In reviewing four studies with fifth and sixth graders and one study with eighth graders, Fennema (1972a) examined reports of the use of various concrete models for teaching different topics appropriate for the designated grade levels. All five studies indicated no significant difference between the group taught by concrete models and the group taught by symbolic models on tests given immediately after the instructional period. One study indicated a difference in favor of the concrete model group three months after instruction.

In summary, the use of appropriate concrete models in the early grades receives support from research. For older learners, either concrete or symbolic models may be equally effective. Other variables may be more important than the particular model selected. Among these variables are whether the principle or skill is meaningfully related to previous learning experiences, the enthusiasm of the teacher for the use of the model or approach and the freedom of the learners to choose from altenative models when called upon to solve a problem or to apply learning in a new situation.

2.11 How effective are manipulative materials in mathematics learning?

As noted in the preceding question, the role of manipulative materials in the learning of mathematics has received considerable attention by researchers in the past 20 years. Evidence indicates that the use of concrete materials appears to be essential in providing a foundation for mathematical ideas, concepts and skills. Reys (1971) suggested that learning theory provides a rationale for using manipulative materials in the sense that learning is based on experience (concrete to abstract), learning is a growth process and is developmental, and learning requires active participation by the learner. Suydam and Weaver (1975) noted that the research increasingly indicates a need to analyze when, with whom, what type and how manipulative materials should be used in mathematics instruction.

2.12 What is the role of hand-held calculators in mathematics instruction?

Five years ago, there were very few small electronic calculators on the market and they were too expensive for common use. However, the increasing availability of hand-held calculators in society, schools and homes has forced educators to seriously consider the role of hand-held calculators in instructional practices. It has been predicted that by 1980 over 80 million calculators will be in use in the United States and that approximately 20 million calculators will be purchased each year. Thus, calculators will become commonplace. The advent of the development of this technology in such a short period of time has not allowed for systematic investigation of how and when calculators can be used most effectively. Many of the initial investigations have been small-scale school trials of calculators, and the results have been variable.

Ladd (1973) reported a study suggesting that a ninth grade course for low-achievers in mathematics, organized into a sequence of short lessons containing problems selected from local businesses, resulted in a significant improvement in mathematics attitudes and achievement. However, the addition of calculators did not increase or decrease the improvement in attitude or achievement.

Mastbaum (1969) conducted a study involving 87 seventh grade students and 84 eighth grade low-achieving students. The results indicated that students could learn to use calculators to solve simple computational problems; however, this ability did not transfer to solving problems in non-calculator activities. He also found that calculators did not significantly improve attitude; nor did they increase mathematical ability, non-calculator computation skills, mastery of mathematical concepts or ability to solve mathematical problems. Keoughard and Burke (1967) reported that groups using calculators in grades 11 and 12 achieved significantly higher than did non-calculator groups.

Hutton (1976) investigated the effects of the use of calculators in ninth grade algebra classes. There was no significant difference in achievement or attitudes between groups using calculators as teaching aids, student tools, or non-calculator groups. The groups using calculators as teaching aids and student tools scored significantly higher on the Math-Fun vs. Dull Scale. The teachers and students participating in this study viewed the calculators with enthusiasm.

A study by Lenhard (1976) investigated the use of calculators in classes ranging from seventh and eighth grade mathematics to Algebra II and Trigonometry. There were no significant differences between experimental and central groups on test scores, concept and computation errors, attitude or time. Borden (1976) found that sixth grade students who used a hand-held calculator as a learning agent gained significantly in achievement of decimal concepts, as demonstrated on a paper-and-pencil test not using the calculator. Students who did not use hand-held calculators had a significantly negative change in attitude toward mathematics.

Jamski (1976) reported a study on the effects of hand-held calculators on the achievement of seventh grade students when they algorithmically converted a simplified rational, decimal or percent to an equivalent form. The calculator groups scored significantly higher on tasks involving the conversion from simplified rational to decimal form.

A study by Jones (1976) compared the effects of the hand-held calculator on mathematics achievement, attitude and self-concept. The results indicated that using calculators was more effective in total mathematics achievement, computation and concepts than using pencil and paper only.

Sutherlin (1976) investigated calculators' effects on fifth and sixth grade students' acquisition of decimal estimation skills. Both the non-calculator and calculator groups studied decimal operations with a stray analysis on estimation techniques. Both groups gained in estimation skills, but the gains did not differentiate the treatments. Calculators were judged by teachers in the study to be durable and easy to store. Sutherlin reported that calculators resulted in quieter classrooms generally, except for enthusiastic sharing of discoveries. The calculators provided incentives to attempt difficult problems and appeared to support trial and error methods of problem solving.

The effectiveness of hand-held calculators in eighth grade fundamental mathematics classes was investigated by Vaughn (1976). The calculator groups achieved significantly higher than the non-calculator groups on decimals and percent. There was no significant difference with respect to attitude, except for the difference normally attributed to achievement.

Even though pertinent research studies are somewhat limited and their findings variable, the increasing automation in society at large, as well as the increasing availability of hand-held calculators, will force educators to adjust curriculum and teaching methods to accommodate calculator use. It is unclear at this point what changes are forthcoming in the next decade.

The National Advisory Committee on Mathematical Education (NACOME) report (1975, p. 41) suggests that if mathematics educators are to take full advantage of new technological capabilities, they will need to make, at a minimum, the following changes.

1. The elementary school curriculum should be restructured to allow an earlier introduction to and greater emphasis on decimal fractions and the metric systems; this will provide an opportunity to avoid the more difficult procedures involving common fractions.

2. Experimentation with decimals should lead to work with concepts and operations involving negative integers, exponents, square roots and scientific notation.

3. The assumption that prerequisites are needed for the study of certain mathematical concepts and applications will need to be changed. This should allow many low-achievers to overcome negative attitudes and to study such topics as probability, statistics, functions, coordinate geometry, etc.

4. The calculator should provide a de-emphasis on mechanical aspects of mathematics and thus allow greater attention to problem solving and applications. The availability of the calculator will force closer attention to analyzing problems, determining appropriate calculations and interpreting results for their reasonableness.

5. The present standards of mathematical achievement will need to be changed to accommodate the use of calculators.

The *Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics* (1976, p. 13) notes that most of the usual expectations about the learning of counting numeration and number processes will remain integral components of a calculator-oriented curriculum. The report emphasizes the importance of "number sense," estimation and understanding our number system and its uses, including

1. the ability to multiply and add one-digit numbers nearly automatically;

2. a firm understanding of various meanings of fractions (i.e., part of wholes, ratios, division) and equivalent fractions;

3. a firm understanding of our numeration system, including representation of arbitarily large, small, whole, or decimal numbers; and

4. experience with algorithms and systematic ways to compute the sums, difference, product, or quotient of any two reasonable numbers.

This conference report also emphasizes the importance of training and retraining teachers to respond to calculators and calculator-influenced curriculum materials. It also makes recommendations for needed research and development related to hand-held calculators in school mathematics.

Despite the obvious promise calculators hold for enriching mathematics instruction, there are many unanswered questions. The NACOME report (1975, p. 43) raised the following questions.

1. When and how should calculators be introduced so that they not block understanding and skill with operations and algorithms?

2. Will ready access to calculators facilitate or discourage student memory of basic facts?

3. For which mathematical procedures is practice with step-by-step paper and pencil calculation essential to thorough understanding and retention?

4. What types of calculator design—machine logic and display—are optimal for various school uses?

5. What special types of curriculum materials are needed to exploit the classroom impact of calculators?

6. How does calculator availability affect instructional emphasis, curriculum organization and student learning styles in higher level secondary mathematics subjects such as algebra, geometry, trigonometry and calculus?

According to Rudnick and Krulik (1976), the mini-calculator is creating a greater controversy than either new mathematics or metrics. Sullivan (1976) states that mathematics teachers are skeptical about mechanical miracles that will revolutionize instruction because they've been disappointed many times before. Ockenga (1976) states that calculators are particularly useful with the more able students who can use calculators as a tool for skills they have already mastered, and can help them advance more rapidly in the mathematics curriculum. Calculators might also help the less able students in grasping mathematical concepts.

In an experiment conducted by Barrett (1974), aimed at using calculators to reinforce basic mathematics skills among heterogeneous grouping of low, medium and high ability students, results showed better test scores, higher student interest and greater comprehension. Barrett noted that having a calculator present was almost like having another person in the room working with him. Schnur and Lang (1976) investigated whether elementary students achieve greater mathematical computational ability through controlled use of calculators, and whether transfer is made to situations in which the calculator is not used. Results showed that the experimental calculator group gained significantly more than the control group.

Many people seem to fear that calculator use will impair students' ability to perform paper and pencil algorithms. Hopkins (1976) notes that when students use paper and pencil, it is possible for them to learn computation skills in a merely mechanical way. However, he notes that calculator use neither alleviates nor increases the danger of students learning how to compute without knowing what they are doing.

The Mathematics Education Action Research Center, in conjunction with the West Chester, Pennsylvania, public schools (Rudnick and Krulik, 1976), initiated a controlled experiment with 600 seventh grade students. The experiment was designed to measure the effect of calculators on students' total mathematics achievement and on their ability to perform pencil and paper basic skills. The experimental group was given calculators during mathematics class each day. Results indicated that there were no significant statistical differences between the control and experimental groups. There was only a slight difference, favoring the experimental groups.

One argument for using calculators is the fact that calculators can be used to solve problems previously considered too time-consuming or impractical to be solved with paper and pencil. Shumway (1976) suggests that because of their simplicity and speed, calculators allow children to work with various functions and numbers of all sizes with a frequency never before possible. He emphasizes that intuitive number sense will be greatly facilitated by extensive, continuous and early experience with numbers and their properties.

Another argument for using calculators involves the notion that calculators increase the speed and accuracy with which children can do calculations This allows much more available time for teaching mathematics in depth, and it places the emphasis on when and what operations are to be used.

Those against using calculators in mathematics classrooms argue that attitudes about mathematics might change—that students might consider mathematics as an exercise in pushing buttons on a black box. To rebuke this, Shumway (1976) notes that people's uses of mathematics will increase astronomically with calculators. Sullivan (1976) suggests that the calculator encourages students to investigate topics ordinarily out of bounds because of computational complexity.

Another argument against using calculators is that paper and pencil algorithms would still be necessary basic skills, because calculators can never be everywhere. In rebuttal of this argument, Judd (1976) suggests that introducing calculators in the classroom will result in shifts of emphasis in both subject matter and skill development. Computation skill will still be taught, but with greater emphasis on facts and decreased emphasis on algorithmic computation and drill. Mullish (1974) predicts that grade school and college students will probably wear miniaturized five function calculators on their waists within the next decade. He also predicts that desks in colleges and grade schools will have scientific calculators built into them.

A further argument against using calculators is that they might destroy all motivation for learning basic facts, producing children who could do only simple calculations on calculators. In rebuttal, Mullish (1974) argues that if a child is given a bicycle to ride to school, he/she certainly doesn't forget how to walk. But the fun of riding a bicycle to school is a stimulating experience, and Mullish feels it is the same with calculators. They serve as motivators, but they shouldn't make students forget basic skills.

Another argument against using calculators is that parents are opposed to their use and feel that schools have failed in the teaching of basic skills. However, Gawronski and Coblentz (1976) point out that calculators are small, almost silent, accurate, easy to use and very fast. The basic skills needed by shoppers in supermarkets, adults balancing checkbooks, students in classrooms and at home all may be aided by using calculators to help solve problems. Shumway (1976) implies that calculators provide experience with the only practical algorithms used in society today. Business, professions and families are not going to carry out extensive calculations without the use of calculators, he says. According to Judd (1976), the calculator will open the door to "the real world." The subject matter studies will be more related to advertisement, monetary exchange and dynamics of everyday living. With the use of the calculator, students can manage a stock portfolio, investigate pollution problems, fill out tax forms, verify bills, orders and invoices and other such activities.

A final agrument against using calculators relates to the cost, which may be prohibitive. Schur and Lang (1976) note that the expenditure of a few hundred dollars for calculators—tools that are highly motivational and provide immediate knowledge of results—is comparable to the purchase of other instructional materials. Another anit-calculator argument deals with the fact that batteries lose their charge and wear out. Some argue that it is foolish to depend on batteries for computational arithmetic. That is, if a child should lose his/her calculator or if the battery should run down, he/she would not be able to do the basic computations. Hopkins (1976) addresses this issue when he argues that if you take the paper and pencil away from students today, or if they forget their pencils, they cannot calculate then either.

The debate likely will continue for years to come on how and when to use calculators in the mathematics classroom. At this point, little conclusive research has been done, but opinions and feelings are strong, indicating that more research will likely be forthcoming.

3. DIFFERENTIATING INSTRUCTION

3.1 How effective are grouping procedures in terms of providing for individual differences?

Students in any classroom differ in intelligence, attitude, achievement and aptitude. It has long been common practice in elementary schools to provide for individual differences in reading by intra-class grouping. However, grouping in mathematics classrooms is less commonplace. These practices are certainly not based on the fact that individual differences in mathematics achievement do not exist. Thomas and Thomas (1965), in a study of individual differences, reported that the differences increase from elementary school to high school. For example, in a typical fifth grade classroom, mathematics achievement based on natural norms range from third to ninth grade. In a typical ninth grade class, the scores range from third grade to grade 12 or above. The researchers conclude that it seems hardly possible that a lesson designed for an average student in a given grade could possibly meet the needs of a typical collection of students.

The effectiveness of grouping procedures in mathematics classrooms is somewhat clouded by the fact that such grouping is based on two different criteria: ability and achievement. At the elementary level, studies by Provus (1960) and Below and Raddell (1963) suggest that homogenous grouping is especially effective for students with high IQs. However, Below and Ruddell noted that decreased range grouping was more effective than homogenous or heterogenous grouping. Savard (1960) reported that decreased range grouping tended to be more effective for lower-ability students than for upper-ability students.

Thelen (1963) noted that teachers seldom agree on who the good students are in a given classroom. He cited the comprehensive review of research on ability grouping over the preceding 50 years by Ekstrom. Ekstrom reported a lack of conclusiveness in the research and found that student achievement was high in the ability grouped classes only about one third of the time. He suggested that the inconclusiveness was due to teachers not knowing how to adapt their teaching methods to accommodate particular groups of students. Thelen indicated it appeared that most teachers tend to teach in the same manner to both high or low ability students. He further concluded that teachers generally perceive "good" classes as being brighter than more "difficult" classes, when in fact a comparison of IQ scores may reveal no difference.

Studies of grouping procedures at elementary and secondary levels based on achievement by Koontz (1961), Mahler (1962), Mikkelson (1963), Baily (1968), Alan (1969) and Eddleman (1972) found no significant differences in achievement between students grouped homogeneously and those grouped heterogeneously. Dewar (1963) concluded that intra class grouping benefited high and low achieving groups more than did total class instruction.

Bierden (1968, 1970) and Mortlock (1970) reported that a combination of whole class instruction and flexible intraclass grouping based on achievement of specific objectives resulted in significant gains in computation skills, concept knowledge and attitudes, as well as a reduction in anxiety.

Broussard (1971) reported the fourth grade students in inner-city schools given individually prescribed work with small group discussion, large group activities, teacher-led discussion and independent study achieved significantly higher skills and concepts than students using traditional text materials or the intra class grouping method.

Bloom, Hasting and Madaus (1971) have proposed that mastery learning should be an important component of individualized instruction. They point out that in a typical classroom all students receive essentially the same instructional treatment with the same amount of class time allotted to complete assignments. They argue that students should receive individualized instruction based on the needs and interests of each student with as much time as necessary given to master the material. Feedback and corrective procedures are emphasized as a mechanism to help students learn those objectives not mastered. Bloom's model for mastery learning is designed to produce students who achieve at a high level, even low achievers.

In the past several years, mastery learning has been the subject of considerable research interest. Good reviews of this research have been conducted by Block (1971, 1974). Collins (1971) studied the effect of a combination of mastery learning strategies on the achievement of eighth grade mathematics classes. He found that 80 percent of the students who were given tests of instructional objectives, test problems to review (based as the objectives) and extra assistance on topics not mastered attained the mastery criterion of grades A or B. He also noted that grades of D and F were practically eliminated.

3.2 How effective are individualized mathematics programs?

In recent years there has been an increased interest in individualizing instruction to better meet the needs of students. Many efforts by classroom teachers to use more flexible grouping procedures and a variety of instructional materials have proven quite successful. More systematic approaches have tended to use some form of self-paced instruction. Self-paced instruction usually involves the use of sequenced learning packets or units with pretests and posttests. Successful completion of the packet and the posttest are usually prerequisites for moving to the next unit or packet. Some programs have incorporated media, but the most typical approach involves small group or individual work from packets, textbooks or worksheets. The teacher in most self-paced programs becomes a manager, tutor, record keeper and monitor.

The more widely used self-paced programs include Individually Prescribed Instruction (IPI), Program for Learning in Accordance with Needs (PLAN), and Individually Guided Education (IGE). In a progress report on IPI programs (1969), it was reported that IPI pupils did as well as non-IPI students on standard achievement tests. Fisher (1967) found no significant differences on achievement of students exposed to IPI, programmed learning or traditional classroom instruction. Schumaker (1972) reported no significant differences in mathematics achievement, study habits or study attitudes between seventh graders involved in IPI or non-IPI programs. Fielder (1971) found that non-IPI students in grades three and six achieved better in IPI. Clough (1971) reported higher achievement and more positive attitudes for IPI students.

Schoen (1976) reviewed a variety of studies comparing self-paced programs to traditional programs. All the studies included in his review met the following four criteria: (1) Comparison groups were equivalent before treatment on the basis of random assignment or matching on the basis of several variables. Otherwise, analyses of co-variance was used or differences between pretest and posttest scores were used as criterion measures. (2) The length of the studies was one semester or longer. (3) Samples were predominantly white middle class students. (4) Criterion measures were standard arithmetic achievement tests or subtests.

Schoen (1976), in a review of 76 studies meeting these criteria, found that over half the studies reveal no significant differences between self-paced (SPI) and traditional instruction (TI). The results indicated SPI was most effective with children in grades K-3 and least effective in grades 4-9. Schoen noted that some questions were raised about the appropriateness of standardized achievement tests as criterion measures. However, he noted that studies using criterion-referenced tests designed to measure the objectives of SPI programs support the conclusions of this review.

Schoen noted that 40 of 55 studies revealed no significant differences on affective criteria. The SPI group were favored over TI groups only at the primary level. He also concluded that entering ability level and self-motivation were the best predictors of success in SPI. Overall, high ability students did about as well in SPI as in TI. However, low ability students were not very effective in SPI programs. In terms of cost, Schoen reported that SPI programs cost more than TI. Sources of increased cost were attributed to materials, teacher aides, teacher-training and occasional building renovation and computer use.

In recent years Individually Guided Education (IGE) has received considerable attention as a method for organizing instruction in elementary schools. IGE is based on a system developed by Klausmeier and others. (Klausmeier (1972); Klausmeier, Hubert, Rossmiller, and Saily (1976))

3.3 What effects have innovative approaches to instruction had on student attitudes?

Over the years research has focused on whether or not innovative approaches to instruction (such as student-centered, discovery or other experimental approaches) have resulted in more positive attitudes and greater achievement than traditional approaches. Aiken (1976) summarized 16 such studies and arrived at the following conclusions.

1. Innovative programs show no greater effect on attitudes than do traditional programs.
2. "Continuous progress" classes have no different effect on attitudes than regular classes.
3. Discovery approaches are not superior to expository methods with respect to attitudes.
4. Attitude is not affected by either follow-up instructions or by flexible scheduling as opposed to traditional instruction.
5. An individualized approach may or may not affect attitudes.
6. Special topics of study have a more positive effect or a more negative effect than other topics.

3.4 What does research on self-concept imply for school programs?

A recent report by Shavelson, Hubner and Stanton (1976) concluded that although there has been a sharp increase in the number of studies on self-concept over the past ten years, there are research problems in this area. Taken individually, the studies provide insights into factors that motivate students—both in the school environment and outside the school environment—into alternative behaviors that may enhance students' self-concepts. The body of research on self-concept as a whole, however, is in need of some standardization. To date, the studies lack a focus that would result from an agreed-upon definition of self-concept, lack proper validations of the interpretations of the self-concept instruments and lack empirical data on the equivalence of the self-concept instruments.

3.5 How has research in self-reliant thinking affected mathematics instruction in recent years?

Erich Wittmann (1971) succinctly posed the problem of dualism between self-reliant thinking and instruction in mathematics. He said that,

> *"Students are only able to acquire a positive attitude to mathematics if they are given sufficient opportunities to practice self-reliant actions and if they receive appropriate help."*

He considers two possible explanations for this existing dualism. He said

1. education in self-reliant thinking "may be achieved only through long training within a suitably coordinated framework and relies heavily on teachers who possess the appropriate attitudes," and

2. those who investigated "productive and creative thinking in the first half of this century concentrated on the major achievements of human intelligence and over-simplified the conditions in which they originate." (Wittmann, 1971, p. 244)

3.6 What does research indicate concerning learning disabilities and mathematics instruction?

According to some estimates (Mackie, 1969) learning disabilities represent the largest category of exceptionality. Myklebust and Bosher (1969) indicate that typical school populations may contain as many as 15 percent learning disabled students. The identification and treatment of learning disabilities are modern phenomena. Research during 1940s and 1950s began to recognize that children were displaying learning problems which do not fit into earlier established classifications of exceptionality. In 1963 the *Association for Children with Learning Disabilities* was formed, and by the early 1970s the area of learning disabilities was clearly established. Despite the recency of the growth in attention and focus, learning disabilities remains a difficult area to adequately define in a concise manner.

A survey of various definitions of learning disabilities yielded combinations of isolated problem areas such as disorders of verbal communications, visual-motor integration, motor activity, emotionality, perception, symbolization, attention, memory, etc. Unfortunately for practitioners, most literature in the field did not attempt to relate specifically to subject matter areas.

Lerner (1976) suggested that problem areas of dyscalculative children included disturbance of spatial relationships; disturbances of visual-perception and visual-motor association; poor sense of body image; poor sense of time and direction and arithmetic problems caused or compounded by reading handicaps.

In general, not a great deal is known about the nature of learning disabilities in mathematics, but research literature revealed the following symptoms and problem areas. [Johnson and Myklebust (1967); Kaliski (1967); Crelchley (1970); Frostig and Maslow (1973); Bartel (1975)]

1. Inability to associate numbers with numerals
2. Transposition of numbers—for example 14 to 41
3. Inability to recognize part-whole relationships
4. Auditory memory problems in oral drill
5. Motor pattern difficulty in writing numerals
6. Inability to understand the meaning of process signs
7. Difficulty in perceiving patterns and sequences
8. Visual-spatial problems with distinguishing differences in size, shape, amounts or lengths
9. Problems with transfer of one process to another (perseveration)
10. Inability to visually recognize numerals

11. Failure to learn to tell time
12. Hyperactivity resulting in impulsive, careless and non-analytic performances
13. Left-right confusion
14. Verbal expression difficulties

Freidus (1966) offered four major ways to help brain damaged children who have problems with arithmetic. These children need help in (1) receiving sensory information reliability, (2) processing the information received, (3) organizing and executing a response appropriate to the perceived meaning and (4) establishing and using some form of self-correcting monitoring habit to determine whether or not the expected behavior is appropriate or inappropriate.

The literature provided the following guidelines for teaching the learning disabled. [Frostig and Maslow (1973); Johnson and Myklebust (1967); Kaliski (1967); Bartel (1975); Ashlock and Humphrey (1976)].

1. Body movement and manipulative objects should be used to develop an understanding of changes in process.
2. Concrete materials should be used to facilitate numerical thinking.
3. The mastery of counting skills should be emphasized.
4. Instruction should integrate counting with one-to-one correspondence maintaining auditory senses of numerals and relating symbols to quantity.
5. Instruction should emphasize relationship between physical objects such as quantity, shape and size.
6. Finger counting and computing should not be discouraged.
7. Careful attention should be given to language.

3.7 What is the role of diagnosis in mathematics instruction?

The role of diagnosing computational errors has long been a concern of mathematics educators. Brueckner (1930, 1938) reported the results of his extensive efforts to identify the types of errors commonly made in computation. His investigations stressed the importance of analyzing written work with some attention given to the need for student interviews. Buswell and John (1926) conducted a study on the four fundamental operations. Their approach emphasized skillful questioning and observation of students as the students worked.

The results of this and other research through the years have focused attention on the importance of diagnosing the strengths and weaknesses of learners in the mathematics classroom. In the case of weaknesses, it is important to identify the causes and provide appropriate remediation. Bernstein (1959), in a review of the research on remedial mathematics, noted that every reported investigation involved instruction based on individual diagnosis. *The Proceeding of the Third Natural Conference on Remedial Mathematics* (1976) emphasized the vital role of diagnosis and diagnostic procedures related to providing corrective or remedial instruction. Holzman and Boes (1973), in a study conducted for the United States Office of Education, found eight common characteristics of successful compensatory education programs included (1) clear objectives stated in measurable terms and supported by instructional techniques and materials closely related to the objectives; (2) attention to individual needs, **including diagnosis and individualized instructional plans** and (3) a structured program approach that stressed sequential order and activities with frequent and immediate feedback.

The criteria for identifying a successful program was student achievement, attendance, positive self-concept and fulfillment of physical needs.

Scott (1970) reported that low achieving seventh graders using programmed materials appropriate for diagnosed needs made significantly greater gains scores in computation than students in regular classrooms. Fennell (1973) found that small group instruction based on diagnostic, prescriptive and goal-referenced strategies required less time for mastery than instruction based on traditional approaches. However, there were no significant differences on achievement or attitude measures. Dunlap (1971) reported no significant differences in achievement of fourth grade students on standardized tests between groups using diagnostic activities or textbook materials. However, the diagnostic activities groups scored higher on the concept section of the experiment's test and the textbook groups scored higher on the computation.

Ashlock (1976, p. 7) provided the following guidelines for diagnosis: (1) Be accepting, (2) Collect data—do not instruct, (3) Be thorough and (4) Look for patterns. Ashlock (1976, pp. 8-9) also provided the following guidelines for remediation.

1. Encourage self-appraisal by the child.
2. Make sure the child has the goals of instruction clearly in mind.
3. Let the child state his/her understanding of a concept in his/her own language.
4. Protect and strengthen the child's self-image.
5. Structure instruction in small steps.
6. Select practice activities which provide immediate confirmation.
7. Spread practice time over several short periods.
8. Use a great variety of instructional procedures and activities.
9. Provide the child with a means to observe his/her progress.
10. Choose instructional procedures that differ from the way the child was previously taught.
11. Encourage a child to use aids as long as they are of value.
12. Let the child choose from materials available.
13. Emphasize underlying concepts and procedures.
14. Emphasize ideas which help the child organize what he/she learns.
15. Stress the ability to estimate.

Although there has been only a limited number of well designed studies conducted to assess the diagnostic approach to instruction, the *Proceeding of the Third National Conference on*

Remedial Mathematics (1976) emphasized that diagnosis should not be limited to slow learners or under-achievers but should be directed toward all children. Likewise, the report urged that diagnosis and prescriptive teaching or corrective teaching be developed for both clinic and classroom application, so that these techniques become available to all teachers.

3.8 What are some procedures for identifying mathematically gifted students?

Identification of gifted mathematics students consists of the application of a set of procedures designed to obtain evidence that an individual does or does not belong to that population of gifted. Selection of procedures is guided by the conceptual definition of mathematical giftedness. Also affecting the choice of procedures are other variables such as finances available for the selection process.

Fully developed selection programs usually separate the process into two phases—screening and final selection.

A screening phase is usually employed, since the cost of final selection procedures is often great in terms of money and time of highly trained personnel. During the screening phase, persons with limited training can use fairly simple procedures such as group tests, grades and teacher nominations to select for further study the individuals being considered as gifted.

The final selection phase generally consists of an intensive, comprehensive study of each individual referred by the screening process to determine if he/she belongs to the gifted population.

In planning identification programs, it is necessary first to define mathematical giftedness. Definitions may vary from community to community depending on what levels of ability and talents are to be included, on allowances to be made for environmental deprivations and on other factors (Martinson, 1974). Once giftedness has been defined a set of procedures is decided upon. Then considerations such as financial resources and availability of trained personnel will influence the choice of selection procedures.

There are several classes of procedures commonly used to identify gifted students. Traditional standardized IQ tests have been the most frequently used among the identification procedures. There are a variety of individual and group standardized tests—the individually administered test being viewed as either "important" or "essential" by 90 percent of 204 authorities polled in a study by Operations Research Incorporated (1971).

Measures of high achievement is another class of procedures. Standardized tests of mathematical achievement, grades in mathematics classes and recognition of achievement such as special honors or scholarships are examples in this class.

Teacher nominations and observations are often used to identify gifted students. Checklists or ranking schemes are useful to teachers for recommending these students.

Non-traditional measures of intellectual abilities include aspects of divergent thinking, evaluation and many aspects of convergent thinking which are not adequately represented in the traditional IQ tests. This class includes various types of creativity tests.

Multiple-identification procedures are more often the rule than the exception (Gloss, 1969; Jackson, 1971). Any measurement technique involves some error; therefore, multiple techniques are often used to compensate for error introduced by a single measurement. The selection of specific procedures depends on the definition of mathematical giftedness and on research of how well given procedures identify gifted students. For example, while it might be reasonable to expect that scores on tests of specific mathematical ability factors would surpass general intelligence test scores as predictors of success in mathematics, actual findings have frequently been otherwise (Aiken, 1973). Also, several research results indicate that previous mathematics grades are good predictors of later achievement in mathematics (Wick, 1965).

Recommendations of procedures to identify gifted students abound in the literature. However, very few results have been reported in the area of **mathematical** giftedness. While application of a set of general procedures to identify gifted students is useful in finding mathematically gifted, more research is needed to assist in designing improved methods to identify mathematically gifted students.

4. METHODS OF INSTRUCTION

4.1 Is conservation of number necessary for understanding addition and substraction?

As mentioned elsewhere in this paper much research has been undertaken to explore the validity of the stages of development proposed by Piaget and the effects of accelerating the development of these stages. Less research has focused on specific implications for classroom practice.

Several researchers have reported results relating conservation of number to counting, addition and subtraction. Almy, Chittenden and Miller (1966) noted that early conservers achieve at a higher level in beginning arithmetic tasks than those who demonstrate conservation at a later age. Robinson (1968) found relationships between conservation, seriation and classification ability and first grade achievement. Steffe (1967) reported that ability to conserve found in children entering first grade was positively related to success in learning addition facts and in solving addition problems. In a companion study LeBlanc (1968) reported the same results relative to subtraction.

Baker and Sullivan (1970) pointed out that assessing a child's development before attempting work with operations assumes a clear-cut line between conservation and non-conservation. Their study suggests that in testing for conservation ability, both task variables such as types of objects used and size of sets, play important roles in the responses of children.

Van Engen (1971) provided children with numbers to add such as two and three and with conservation problems dealing with the combination of two sets. Confronted with piles of candy, one with two pieces and the other with three pieces, each child was asked whether he/she would prefer having the two piles, or one pile created by combining the two piles, or whether there is a difference. Only one child in the 100 first-graders tested could not add two and three correctly; however, only about 50 percent were able to solve the conservation problem.

Almy (1970), who reported ambiguous results, concluded that a child's development may be unevenly paced going both forward and backward rather than in an ever-advancing sequence. Moreover, to compound the problem, he concluded that individual children differ greatly in cognitive level, attitudes, interests and concerns.

In a recent study Npiangu and Gentile (1975) grouped homogeneously 116 children of ages four to six on the basis of a number conservation pretest, then assigned them randomly to two groups. The experimental group was taught arithmetic concepts while the control group played a game. Posttest scores showed that although training did have an effect, it did not have a differential effect on conservers and non-conservers. The researchers concluded that conservation may not be a necessary condition for mathematical understanding. They suggested that it is better to think of aspects of arithmetic and number conservation as concepts that develop together and to direct efforts toward evaluating what arithmetic concepts a child knows when designing formal instruction.

After summarizing studies on the implications of Piaget's work to the classroom Callahan and Glennon (1975, p. 30) observed

> The evidence would suggest to the teacher that data relating Piagetian theory to school mathematics are still quite meager and tentative. The arguments for an isomorphism of mental processes between his genetic development theory and the systematic learning of conceptual mathematics by children in schools get their primary strength from a prior claim rather than empirical evidence.

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4.2 Should children be allowed to count when finding answers?

Brownell (1928) listed four levels of maturity in children's responses to number facts: (1) counting, (2) partial counting, (3) grouping and (4) meaningful habituation. In the early phases of instruction the child should be allowed, even encouraged, to count and group in order to find answers. The goal at more advanced levels should be meaningful habituation, that is, automatic responses to number facts. Brownell and Chazal (1935) suggested that teachers should not expect children to respond at a mature level initially. Progress toward mature thought processes is slow, and the progress from immature responses to true mastery proceeds at different rates for different children.

In their research on the process of enumeration, Beckwith and Restle (1966) pointed out that children from seven to 10 years of age are sensitive to visual organization when counting. Even when a child enumerates one by one he/she may count quickly in one group, pause to organize his/her results, and then proceed to the next group. The ability to separate the task into manageable units is an important part of performing a long serial task.

Sauls and Beeson (1976) found that accuracy was about the same for counters and noncounters in the fourth grade. About 62 percent of their subjects used counting for addition and subtraction.

Wheatley (1976) concluded that fourth grade children who use their fingers when performing column addition are slower than those who know the basic facts of addition. His study indicated that 46 percent of the children involved used finger counting.

The teacher should feed confident that in the early learning phases counting is an acceptable method for finding answers. It is the teacher's responsibility to guide the child into more mature patterns of behavior as the child gains experience.

4.3 Are addition and subtraction facts all of the same difficulty level?

Suydam and Weaver (1975) surveyed the results of a number of studies and drew the following generalizations.

1. Addition combinations are easier to learn than subtraction combinations.
2. Reversing an addition combination produces a combination of comparable difficulty.
3. The size of the addends rather than the size of the sum is more important when determining difficulty.
4. Combinations in which "1" is added to a number are the easiest in addition. Combinations in which "1" or "2" is the difference are the least difficult in subtraction.
5. A common addend produces combinations of similar, but not equal, difficulty.

Using these conclusions teachers should be able to sequence instruction in such a way as to proceed along a hierarchy of difficulty levels for addition and subtraction facts.

4.4 What factors are related to the difficulty level of open addition and subtraction sentences?

Some research is now available regarding the relative difficulty in solving open sentences such as ▱ + 4 = 7; 8 - ▱ = 2; ▱ = 2 + 3, etc.

Weaver (1971, 1972, 1973) used items which allowed him to examine the effect on performance of the operation, placement of the variable and symmetric forms of sentences. His results included the following.

1. Pupils had less difficulty with sentences of the form a+b=▱ and c–a=▱ than with sentences of the form ▱=a+b.

2. Third graders performed better than second graders, and second graders performed better than first graders.

3. On each grade level subtraction sentences were more difficult than addition sentences.

4. Open sentences with the placeholder in the initial position, e.g. ▱–b=a or c=▱–b, were more difficult than any other type.

In summary, Weaver's studies indicate that grade level, operation and position of the placeholder appear to be related to difficulty. The research results also suggested the likelihood of interactions between and among the various factors studied.

Using sentences of the form N+b=c, a+N=c, N–b=c, and a–N=c, Grouws (1972) found a significant difference in the difficulty of solving sentences involving basic facts with sums from 10 to 18 and sentences with addends and sums between 20 and 100. Hence, magnitude of the numbers appears to affect the difficulty of solution.

Howlett (1973) reported that age and intellectual stage of development appear to be related to ability to solve open addition statements.

Peck and Jenks (1976) observed two strategies for solving missing addend problems—counting and memory. Pupils who used counting were able to solve similar problems asked concretely, whereas pupils who used memory were not able to do so. Conservation did not seem to be a factor in the solution of missing addend problems.

Based on conclusions of researchers, it appears that in planning for children to solve open sentences of different types teachers should provide practice in problems of a variety of forms involving both addition and subtraction. They should also take into consideration the age, grade and intellectual level of the child when setting expectations of success.

4.5 What is the role of materials in developing addition and subtraction algorithms?

Ekman (1967) studied the relative effectiveness of three modes of teaching addition and subtraction algorithms to third grade children: (1) student manipulation of cardboard disks before development of the algorithm, (2) development with pictures before presentation of the algorithm and (3) immediate presentation of the algorithm form. Results indicated that the first group performed better on measures of understanding and transfer. Using materials the first and second groups performed at a higher level on the skill scale immediately after the instructional period; however, there was no significant difference in skill performance in the three groups later on a retention test.

Knaupp (1971) compared two instructional methods and two manipulative models for developing addition and subtraction algorithms. He concluded that teacher-demonstration and student-manipulation methods with sticks or with blocks resulted in signficant achievement.

Trafton (1971) reported that children taught by a general approach, emphasizing basic concepts of subtraction and employing work with the number line prior to development of the decomposition method, did not perform signficantly better, nor have better understanding, than children who were given prolonged instruction in using the algorithm without the general approach.

4.6 Should addition and subtraction facts and algorithms be taught at the same time?

Spencer (1968) reported that introducing addition and subtraction concurrently may produce inter-task interference, but an emphasis on the inverse relationship may facilitate understanding.

Wiles, Romberg and Moser (1972, 1973) studied the relative effectiveness of two sequences designed to teach addition and subtraction algorithms for two-digit whole numbers. In one sequence, instruction on addition and subtraction algorithms was integrated with equal emphasis on the two operations. Regrouping was the integrative skill used to accomplish both carrying and borrowing.

In the other sequence addition instruction was completed before subtraction was begun. In comparing group means of two classes of second graders the researchers found that their results favored the sequence that separated instruction on the algorithms. They also concluded that the addition algorithm is less difficult than the subtraction algorithm; although regrouping is a major difficulty in learning the algorithmm for either operation.

It appears from the research reported that it is more beneficial to develop one operation at a time, allowing children to gain confidence in their ability to understand that operation before proceeding to the inverse operation.

4.7 What kinds of problem situations should be used in developing subtraction?

Gibb (1956) identified three types of subtraction problems: take-away, additive and comparative. A take-away problem asks, "How many are left?"; an additive problem asks, "How many more are needed?"; a comparative problem asks, "How many more are in one set than in another?" In her study Gibb found that second grade children performed best on take-away problems. They found additive problems more difficult and needed more time for their solution. Of greatest difficulty were comparative problems. Gibb reported that children solved the problems with respect to the individual situations rather than by appealing to the basic concept underlying all the applications.

Schell and Burns (1962) also observed the same relative order of difficulty for second grade children. They noted that children's diagrams showed a lack of understanding that solutions are different when interpreted visually. The researchers also pointed out that second and third grade textbooks emphasize the idea of take-away.

Research by Crumley (1956) suggested that children tend to see subtraction as a process of taking away regardless of the teaching method employed.

Coxford (1966) and Osbourne (1967) each reported that a set partition approach emphasizing the relationship between addition and subtraction resulted in better understanding than the take-away method.

It appears that teachers should present the different types of problems in a systematic way if they expect children to develop facility in solving each type. The last two studies noted lend support to the development of set and subset concepts as an important aid to understanding the concept of subtraction and its application in physical situations.

4.8 What method should be used for column addition?

Two common methods of adding a column of numbers are the direct method, adding each number in the order that it appears in the column and the tens method, locating pairs in the column which have a sum of ten. Using fourth grade children, Wheatley (1976) found the direct method to be faster than the tens method; however, he found no difference in the accuracy measures for the two methods. By the end of the instructional period the number of students preferring the tens method had increased, indicating that the method has strong appeal, although it takes more time to perform than the direct method. The researcher observed that even though each student had worked about 350 problems during the time of the experiment, there was no improvement in performance except for treatment effects. He conjectured that performance in column addition exists at some base level and is not easily changed by practice.

4.9 What method should be used for teaching subtraction with renaming?

Two widely used methods for performing two-digit subtraction are called the decomposition algorithm and the equal-additions algorithm. In solving a problem such as 63–27=☐, the student using the decomposition method would say 13–7=6 (ones) and 5–2=3 (tens) to obtain 36; whereas, the pupil using the equal-additions method would say 13–7=6 (ones) and 6–3=3 (tens) to obtain 36.

In a well executed study reported in 1949, Brownell and Moser compared the two algorithms in combination with two instructional procedures described as meaningful and mechanical. Some of their findings concerning initial instruction were

1. Children taught by meaningful decomposition performed better than children taught by mechanical decomposition in terms of understanding and accuracy.
2. Children taught by meaningful equal-additions scored significantly higher than children taught by mechanical equal-additions on measures of understanding.
3. Both equal-additions procedures were superior to mechanical decomposition.
4. Meaningful decomposition was more effective than either equal-additions procedure in terms of understanding and accuracy.

In conclusion the researchers suggested that the choice of algorithm should rest on the specification of desired outcomes.

With increased use of hand-held calculators and other electronic processing devices it seems logical to stress understanding and application rather than computational speed; therefore, the teacher should attempt to select algorithms for subtraction, as well as other operations, which contribute to the child's ability to apply them in meaningful ways in practical situations.

4.10. How should multiplication be developed?

Traditionally, multiplication of whole numbers has been presented through a repeated addition approach, e.g. 4 × 3 = 3 + 3 + 3 + 3. Difficulties in conceptualization occur when the first factor is 0 or 1. Some recent studies have explored the possibility of using other methods for developing the concept of multiplication.

Schell (1964) compared the achievement of third graders who were introduced to multiplication through a rectangular array approach with pupils who were presented with a variety of representations. There was no significant difference in achievement levels of the groups.

Hervey (1966) determined that second grade children were more successful in conceptualizing and solving repeated addition problems than Cartesian-product problems. In comparing different groups' success on Cartesian-product problems she found that high achievers were more successful than low achievers, boys were more successful than girls and pupils with above-average intelligence were more successful than pupils with below average intelligence.

Fennema (1972) presented multiplication as the union of equivalent disjoint sets through a concrete model (Cuisenaire rods) and symbolic representation. Both methods were effective when immediate recall was measured; however, symbolic representation appeared to be more effective when transfer and concept extension were the goals.

In addition to selecting a model to use, there is also the problem of choosing a method of presentation. Fullerton (1955) compared two methods of introducing simple multiplication facts to third grade pupils. One was an inductive approach whereby the children developed facts from word problems, using various procedures. In the other process the same facts were presented to the children, but without their involvement in factual development. A significant difference was noted in favor of the inductive method on measures of immediate recall, transfer and retention.

From the research it appears that several models for multiplication are appropriate for primary children. The Cartesian-product approach seems to be more difficult for young children to understand than the repeated addition approach, but there seems to be no other conclusive evidence as to the superiority of one model over another. The last study mentioned supports the active involvement of the learner in the development of facts which may be one of the more important results reported.

4.11 Is attention to mathematical properties helpful in developing multiplication skills?

The number of basic facts of multiplication that must be memorized is reduced greatly if children are able to apply the commutative, associative, identity and zero properties of multiplication and the distributive property of multiplication over addition. Relatively little research has attempted to assess the usefulness of placing emphasis on these properties.

Hall (1967) reported results on the teaching of selected multiplication facts to third graders which appear to favor emphasizing the commutative property in early work.

Gray (1965) investigated the use of the distributive property of multiplication over addition in beginning work on multiplication with third grade pupils. The children who worked with distributivity scored significantly higher on measures of transfer ability, retention of multiplication achievement and retention of transfer than those who did not. They scored higher, but not significantly so, on a posttest of multiplication achievement. Gray's research indicates that the advantages for meaningful learning may not always be evident immediately after instruction. The benefits are sometimes more clearly observed when comprehension, transfer and retention are measured.

In the development of multiplication algorithms, on the other hand, Hughes and Burns (1975) found significant differences in computation scores for a group using the lattice method over a group using the distributive algorithm in grade four. It is possible that after initial meaningful work with properties, children are able to perform better on algorithms that offer shortcuts to obtaining a correct answer.

4.12 Is the same difficulty level attached to all open multiplication and division sentences?

Grouws and Good (1976) studied the performance of third and fourth grade pupils on single-variable multiplication and division sentences in which all numbers involved were whole numbers that did not exceed 81. The following were included in their observations.

1. There was much progress in solving sentences that had whole number solutions, from the third grade level to the fourth grade level.

2. There was no improvement in performance on sentences with no solutions from one grade level to the next. Actually, third graders out-performed fourth graders on this type of problem.

3. Girls scored higher than boys on solutions in general, but there was no difference associated with sex on the no solution type sentence.

4. High aptitude students achieved on a higher level than low aptitude students.

5. More than 70 percent of multiplication sentences with solutions were solved correctly; however, less than 50 percent of division sentences with solutions were solved correctly.

6. Children arrived at correct answers for more division sentences with no solution than multiplication sentences with no solution.

7. Sentences of the form $a \times b = c$ were more frequently solved correctly than sentences of the form $c = a \times b$.

8. Large differences in performance were associated with the placement of the variable. Of greatest difficulty were sentences in which the variable was in the initial factor position, i.e. $\square \times b = c$ or $c = \square \times b$.

9. Sentences of the form $c = \square \div b$ were solved correctly only 16 percent of the time and sentences of the form $\square \div b = c$ only 37 percent of the time.

If teachers want pupils to achieve facility in solving a variety of types of open sentences it is important to note that such factors as operation, place holder position, placement of the operation to the left or right of the equal sign and whether the problem has a solution may affect performance. To insure greater exposure to difficult types, care should be taken in designing activities and written assignments so that all types of problems receive attention.

4.13 What kinds of problem situations should be used in developing division?

Two types of division situations can be identified:

1. In a measurement problem the number of elements in the original set and the number of elements in each subset is known. The solution consists of finding the number of equivalent subsets. Example: If I have eight pieces of candy and separate them into piles of two pieces, how many piles will I have?

2. In a partition problem the number of elements in the original set and the number of equivalent subsets is known. The solution consists of finding the number of each subset. Example: If I have eight pieces of candy and separate them into four piles, how many pieces will be in each pile?

Gunderson (1953) found that partition problems were more difficult for second grade children to solve than mesurement problems. Hill (1952) reported that upper grade children prefer measurement problems but found no significant difference in their performance on the two types of problems.

Zweng (1964) offered a further breakdown of problem types by differentiation between basic measurement and rate measurement situations, and between basic partition and rate partition situations. A rate situation involves a second set, such as a set of sacks to hold objects to be shared. An example of a rate measurement problem is: If I have eight pieces of candy and put them into sacks by placing two pieces in each sack, how many sacks will I use?

Zweng's observations of the work of second grade children included the following.

1. Partition problems were more difficult than measurement problems.
2. Basic partition problems were more difficult than rate partition problems.
3. Problems with one set of objects were more difficult than problems with two sets of objects.
4. Difference between partition situations using two groups of objects and measurement problems were not significant.
5. Children solved partition problems in two ways—(1) grouping, or assigning all of the elements on the first trial; (2) sharing, or assigning the same number of elements to each subset but not all on the first processing. When using sharing, the children in the study most often did not use a one-to-one sharing but chose on their first assignment a number that was needed for each group.

All of the studies cited indicate that measurement situations are easier for children to solve. In presenting division, the aware teacher will help children learn to deal with partition situations as well as measurement situations.

4.14 Which algorithm should be taught for division, the conventional or the subtractive?

Two algorithms which have been used in the schools are illustrated below.

```
     Conventional or               Subtractive or
 Distributive Algorithm        Greenwood Algorithm

           23
         4/92                     4/92
            8                       40     10
           12                       52
           12                       40     10
                                    12
                                    12      3
                                           23
```

Van Engen and Gibb (1956) compared the two forms and found advantages for each. They concluded that

1. When taught by the subtractive method pupils understood the process or concept of division better than when taught by the conventional method. Children with high ability applied the two algorithms with the same effectiveness; however, children of low ability were more successful when the subtractive method was used.
2. When taught by the conventional method, children had higher problem-solving scores on the type of problem used in the study.

3. Children were able to transfer to new but similar situations more effectively through the subtractive approach.

4. On measures of retention and understanding the two methods appeared to be equally effective.

5. Children were more successful in working with partition situations when using the conventional algorithm and more successful in working with measurement situations when using the subtractive algorithm.

On the basis of their observations the researchers did not recommend converting to the conventional algorithm after the learner had been taught the subractive algorithm, as is the practice in many textbooks.

Scott (1963) studied the effect of using the conventional algorithm for partition situations and subtractive algorithm for measurement situations. His research suggested that

1. Using the two algorithms was not overly difficult for third graders.

2. No more teaching time was required for the two algorithms than for one algorithm.

3. Children who were taught both algorithms indicated a greater understanding of division.

Dilley (1970) studied the two approaches with fourth graders and found little difference in performance and no interaction between socioeconomic level or ability level and the two treatments.

Kratzer and Willoughby (1973) compared the relative effectiveness of a subtractive approach to the Greenwood algorithm with a partitioning approach to the conventional algorithm. For the fourth grade subjects in their study they found

1. The partition approach to the conventional algorithm led to higher achievement on the total set of problems on an immediate test, a four-week retention test and a delayed-retention test.

2. No significant difference in achievement existed between the two approaches on problems similar to those studied in the lessons.

3. On unfamiliar problems the children who studied the partitioning approach scored significantly higher in all three testing situations (immediate, four-week and delayed-retention).

4. There was no significant difference in achievement between the two approaches on an immediate test of verbal problem solving. A significant difference in achievement on the four-week test favored the partition approach, but no difference was found on the delayed-retention test.

The most recent research seems to favor the conventional or distributive approach over the subtractive approach. There appears to be contradictory evidence as to whether the subtractive algorithm is easier than the distributive algorithm for low abiity students. At this time there appears to be no clear answer to the question of whether children should be taught both algorithms.

4.15 What does research indicate about teaching fractions?

A large number of research studies have been reported which relate to the development of fractional concepts and computational skills. The studies cited here were selected as representative of the kinds of research that have been undertaken.

One type of research has attempted to analyze fraction concepts or skills into a hierarchy of less difficult subconcepts and tasks. One such study is by Novillis (1976), who constructed a hierarchy of subconcepts of the fraction concept and a Fraction Concept Test of 68 items designed to test the validity of the sequencing of her dependent subconcepts. She administered the test to almost 300 students in grades four, five and six and found support for 18 of 23 dependencies in the hierarchy. Among subconcepts which appeared prerequisite to others were

1. The ability to associate a fraction with a geometric region and with a set model is prerequisite to
   a. the ability to associate a fraction with a point on a number line.
   b. the ability to compare a fraction in a situation using the given model.
   c. the ability to associate a fraction with the given model in which the number of parts was a multiple of the denominator and the parts were arranged in an array that suggested the denominator.
2. The ability to associate a fraction with a geometric region or a set model having congruent parts was prerequisite to the ability to associate a fraction with a model having non-congruent parts but parts which, in the case of a geometric region, have the same area.

The researcher inferred from some of her results that

1. Pupils are not exposed to a large enough variety of instances of the fraction concept to encourage them to generalize the fraction concept. For example, a child may associate 1/5 with one of five objects without being able to associate it with two of 10 objects.
2. Pupils are not presented with an adequate number of negative instances of the fraction concept. For example, a child may incorrectly associate 1/5 a geometric model in which the parts are neither congruent nor have equal areas.

In another type of study the researcher has attempted to identify effective approaches for teaching operations on fractions. Some examples are included here.

Pigge (1964) compared three methods of developing the addition algorithm for fractions with fifth grade students. Each method involved a different combination of developmental-meaningful instruction and drill. His results indicated that the pupils derived a greater benefit from a developmental-meaningful approach that incorporated demonstrations and explanations, discussions, audiovisual aids and manipulative materials.

Bisio (1971) reported that fifth grade students profited from concrete aids prior to introduction of unlike denominators. He also found that teacher demonstration of the aids was as effective as student manipulation.

Anderson (1966) and Bat-hall (1969) investigated methods of finding the lowest common denominator (LCD). With fifth graders both researchers reported that the factoring approach and a method using rows of equivalent fractions were equally effective for instruction in finding the LCD. Both processes were superior to the inspection approach.

Green (1969) studied several method for teaching multiplication of fractions to fifth grade pupils. She concluded that an approach based on the area of a rectangular region was significantly more effective than associating the concept "of" with multiplication via a part of a region or a set. The results of the study indicated that the idea of a fractional part of a set was not well understood by the subjects in the study.

Brooke (1954) and Capps (1963) compared the common denominator method of dividing fractions with the inversion method. The methods were equally effective as measured by an immediate test, but the results of a retention test favored the common denominator approach. The Capps study indicated that use of the inversion method had a positive effect on achievement in multiplying fractions.

Another type of study has attempted to identify systematic errors made by students in working with fractions. Concerning multiplication of fractions, Romberg (1968) reported that many pupils appear to know the algorithm but make mistakes in reducing their answers to lowest terms. Further, pupils (particularly in modern math classes) do not apply a cancellation process. Lankford (1972) identified another error in multiplication. Many of the students he interviewed first found equivalent fractions with a common denominator and then multiplied the numerators of the fractions and placed the product over the common denominator. The students in the study who made this error apparently did not understand when and why the common denominator process is applied.

Some studies have investigated achievement of elementary school children relative to fraction concepts and skills. Carpenter et. al. (1975; 1976) reported results from the National Assessment of Educational Progress (NAEP) involving children's ability to recognize and use fractions. The data indicated that about two thirds of the nine-year-olds tested could not correctly translate into numerical form the picture of a region with a fractional part shaded. An analysis of their errors suggested a lack of knowledge rather than a misunderstanding of the question. Considering the 13-year-old subjects in the study 65 percent were able to name a fractional part on a simple example, 20 percent were correct on a more complex exercise, 55 percent were able to find a fraction between two common fractions, 19 percent were able to select the fraction closest to 3/16 from a list of fractions, 26 percent were able to select the largest fraction from a list, 65 percent were able to change an improper fraction to mixed form, 42 percent were able to add two fractions and 62 percent were able to multiply two fractions. The researchers stated:

> *"The concept of fraction is difficult to understand and to use. The low performance of 13-year-olds . . . on the fraction exercises appears to be attributable more to this intrinsic difficulty than to a lack of attention to fractions and fraction concepts in the upper elementary grades. Rather than simply increasing the time spent on fractions and operations with them, teachers and curriculum developers might more profitably examine first what aspects of fractions are being emphasized in the curriculum. A thorough development of intuitive ideas might pay great dividends later when pupils are learning how to calculate with fractions," Carpenter, et. al., (1975, p. 442).*

National Assessment data indicated that the subjects in all age groups had difficulty with fractional concepts and with computational tasks. The researchers in their 1976 report made some specific recommendations for instruction which are summarized below.

1. Provide sufficient work with manipulative materials, such as paper models and diagrams, before formal work with the addition algorithm is undertaken.

2. Provide concentrated work on identifying the unit whole in different representations of fractions.

3. Interweave subordinate ideas, such as unit, part-whole relationships, comparing and ordering fractions, equivalent fractions and renaming improper and mixed form fractions, so that they have meaning for the pupil.

4. Give careful attention to concepts related to factors, multiples and renaming, for these ideas are necessary for understanding the notions of common denominator and equivalent fractions.

5. De-emphasize the requirement to reduce fractions to lowest terms until students are competent in performing the computational algorithm.

For additional suggestions from research, the reader is referred to Anderson (1969).

4.16 What effect does the teaching of non-decimal numeration have on understanding our numeration system?

One of the reasons advanced for teaching non-decimal numeration is conjecture that such instruction will improve the child's understanding of our system of numeration.

Diedrich and Glennon (1970) compared three different groups of fourth graders who received instruction in numeration systems. One group studied only base 10, one group studied bases three, five and 10, and one group studied bases three, five, six, 10 and 12. The research results indicated that if the objective of instruction is an improvement in understanding the decimal system, then the study of the decimal system alone is sufficient.

The question of whether other benefits are derived from the study of non-decimal numeration systems remains unanswered.

4.17 Is there agreement on the geometry and measurement concent that should be developed in the elementary school?

In recent times more emphasis has been placed on the study of geometry and measurement topics in elementary school textbooks, curriculum committee reports and curricular program guidelines than in years past. Great diversity exists in the content of the elementary curriculum recommended by various sources.

Golightly (1976) surveyed the geometry and measurement content developed in nine American textbook series and the recommendations from 10 different programs and well-known contributors (such as Piaget, the Nuffield Mathematics Project and the Cambridge Conference Report) from this country and abroad. She found little common agreement; however, after a careful analysis she was able to formulate a list of content objectives most often represented for grades K-3. Under the heading Shape and Form she identified 14 different objectives which dealt with the ability to identify, to classify and to distinguish between different geometric shapes. With respect to the category Spatial Relationships she listed 18 different objectives which involved the ability to recognize two- and three-dimensional shapes, to recognize and distinguish between congruent shapes and similar shapes, to distinguish symmetrical shapes and to see patterns. Under the heading Measurement she identified 14 separate objectives dealing with time, money, length, area, weight, capacity and temperature. The number and the range of the objectives identified for the primary grades alone gives some indication of the scope of the geometry and measurement tasks common to many modern sources. A look at the programs and proposals themselves verifies the contention that there exists a wide diversity in recommended topics. If such a summary of recommended work for the later grades has been undertaken it seems apparent that it must disclose as much, if not more, variety in recommendations for older elementary children.

4.18 What implications for teaching geometry can be drawn from research?

Williford (1972) summarized research studies on elementary school geometry. He concluded that

1. Most young children have a variety of skills related to identifying and matching planar and solid figures, comparing linear measurements and reproducing parallel and perpendicular segments.
2. Pupils taught by modern programs seem to learn more geometry than students in more traditional programs.
3. Children achieve at a higher level when geometry is taught by a teacher rather than by programmed instruction.
4. For kindergarten children numerous concept examples lead to better understanding; however, numerous concept examples do not appear necessary for older children.
5. Manipulative activity using a large number of examples is very important for kindergarten children. There is some evidence that using a large amount of concrete materials is better than using a minimal amount for middle grade pupils.
6. Expository teaching seems better than a discovery approach for short-term retention; however, a discovery approach appears superior to expository methods for long-term retention.
7. There is a significant relationship between geometry achievement and achievement in reading and mathematics in general.
8. Most studies indicate little or no transfer to other areas from instruction in geometry.
9. There is evidence to suggest that pupils' geometric knowledge is significantly related to the knowledge of their teachers. Research that sought to identify factors related to knowledge concluded that the number and the type of previous mathematics courses are related to the geometric knowledge of both preservice and inservice teachers.
10. A number of studies indicated that elementary school children can learn about a wide variety of geometric topics, including concepts of topology, motion geometry, coordinate geometry and simple constructions.

Although a great deal of research has been undertaken since Williford's article was published, his conclusions remain a good summary for the consideration of the concerned teacher.

4.19 What does research say about teaching measurement?

Measurement encompasses a wide variety of topics. Since the nature and the scope of research related to measurement is quite broad, the concern here will be to summarize a few representative studies.

Very young children differ greatly in their acquisition of concepts of time, money, length and liquid measures. Mascho (1961) surveyed first graders and concluded that familiarity with measurement increased as age, socioeconomic level and mental ability increased. Terms used in context were easier for his subjects to recognize. The study suggested that some concepts commonly developed in the first grade were already part of the child's repertoire of knowledge when entering school. Teachers apparently need to examine the makeup of their classes in terms of age, socioeconomic status and mental ability when designing measurement activities.

Carpenter (1975) tested first and second graders on 13 measurement and conservation tasks. Children in the study did not rely predominantly on visual judgments and were able to indicate an understanding of some aspects of measurement. Almost all subjects recognized that greater quantities are associated with a greater number of units and were able to make comparisons accurately based on measurement operations; however, only 25 percent understood fully the importance of selecting a single unit of measurement and only six percent understood the inverse relationship between unit size and number of units.

Carpenter and Lewis (1976) found support for the conclusions of earlier studies that children have great difficulty in solving measurement problems in which quantities are measured with different units of measure. They found that most of the first and second graders they studied realized the effect of changing unit size and had some notion of the inverse relationship between size of unit and number of units at an earlier age than previous studies had indicated. From the results of their study they conjectured that children do not develop the concept of the inverse relationship between unit size and number of units by measurement experiences with different sized units; moreover, they concluded that such manipulations "may tend to reinforce incorrect notions of length." (p. 58)

According to Steffe (1971), for a child to compare successfully the lengths of two polygonal paths (paths composed of line segments) he/she must possess the ability to use the concepts of conservation of length and of transitivity (if A is related to B and B is related to C, then A is related to C). Using Steffe's observations as a basis, Bailey (1974) explored questions related to the following items: (1) concepts that a child must have to measure length in a meaningful way, (2) the relationships of the necessary concepts to each other and (3) the age level at which children usually possess the concepts. He investigated the reasoning used by first, second and third grade children as they attempted to compare lengths of two polygonal paths. His results indicated that

1. The ability to do transitive reasoning is logically necessary to the ability to use the substitution property in working with length relations and is thus necessary for the comparison of the lengths of two polygonal paths.

2. By age eight a majority of children are capable of using the transitive property of length relations.

3. The ability to use both the length of units and the number of units simultaneously to compare the lengths of two polygonal paths is not a characteristic of children in the age groups tested and, therefore, must appear at a later stage of development.

Many reports of research, such as these of Steffe (1971) and Bailey (1974), which attempt to find relationships between Piagetian tasks and measurement tasks usually given in elementary classrooms appear in the literature. Talournis (1975) presented three area conservation tasks and two area measurement tasks to children in grades one through three. The results of the study indicated that

1. Higher scores were achieved on the set of tasks (area conservation or area measurement) which was presented after the other task.

2. A significant correlation existed between area conservation scores and area measurement scores.

3. From grade to grade there was no signficant amount of score increase on the tasks.

Carpenter, et al. (1975) reported results of the measurement portions of the National Assessment of Educational Progress (NAEP) for nine-year-old and for 13-year-old pupils. They summarized their findings as follows.

> The nine-year-olds demonstrated a very tenuous understanding of measurement concepts and had difficulty with all but the simple measurement exercises requiring a one-step solution process. Most nine-year-olds were familiar with only those aspects of measurement likely to be encountered outside the school curriculum—recognizing common units of measure, reading clocks and making simple linear measurements. The 13-year-olds performed much better than the nine-year-olds but still had difficulty with many basic measurement concepts (p. 445).

Specifically both age groups had difficulty with many of the conversion exercises, which involved measurement in the American system; however, since the metric system is being used more and more frequently, the researchers felt that teachers need not be overly concerned with this aspect of the results. The writers felt that teachers **should** be concerned with results which dealt with basic concepts embodied in any measurement system. For instance, most children could read a clock, but children in both age groups had difficulty in solving problems that required calculations with time. Most nine-year-olds could measure length correctly only if a whole number of units was needed and only if the length was shorter than that of the ruler. The 13-year-olds could measure longer distances but had difficulty in dealing with fractional parts of units. Few children in either age group understood basic notions of perimeter, area and volume.

Among recommendations to teachers from the NAEP researchers were the following.

1. Give particular attention to the development of the concept of unit.
2. Provide a wide variety of measurement experiences at all grade levels.
3. Give instruction and practice in
    - measuring lengths that are longer than the measuring instrument;
    - making measurements that involve fractions other than one-half;
    - making indirect measurements of distance;
    - estimating lengths.
4. Provide more problems involving intervals of time rather than merely giving practice in clock reading skills.
5. Use everyday situations that emphasize the notion that measurement is one of the principal ways mathematics is applied in the real world.

4.20 What suggestions for teaching metric measurement are available?

Although little research has been reported on teaching the metric system specifically, various individuals and committees have recommended guidelines for teachers. The ones summarized below were contributed by the Metric Implementation Committee (1974) appointed by the National Council of Teachers of Mathematics. With respect to teaching measurement in general the Committee advises the following.

1. Begin by measuring with a nonstandard unit—such as a popsicle stick—then select a standard unit for children to use in measuring a variety of objects.

2. When larger or smaller units are needed, use multiples and subdivisions of the basic unit.
3. When converting within a measurement system expect mastery only with common units that are close in size.
4. Emphasize the approximate nature of measurement in the physical world.
5. Through measurement, provide motivation for working with fractions, decimals, and arithmetic operations.
6. Use actual units rather than the scale models that are often found in printed materials.

With respect to the metric system the committee recommends the following.

1. Teach the metric system and the American system as separate systems. Do not emphasie the relation between the units of the two systems.
2. Teach the metric ssytem first since it will be the system used predominately in the future.
3. Use the meter as the basic unit of measurement for middle and upper grade pupils. Use the centimeter or a 10 centimeter unit for primary grade children.
4. Emphasize the most commonly used multiples and subdivisions and their respective prefixes.
5. For children and adults who already know both the metric and the American systems, make approximate conversions.

For more details and suggestions for grade placement of skills, the reader is referred to NCTM Metric Implementation Committee's report (1974).

4.21 What do children know and what can they learn about logic?

Critical thinking depends on making logical inferences, recognizing fallacies and identifying inconsistencies among statements. Some different approaches have been taken by researchers to determine to what extent elementary children have developed logically correct thinking patterns and to what extent they can learn elements of logic.

Hill (1961) reported that children of ages six through eight can draw valid conclusions from premises. She concluded that the growth of logical thinking was gradual but somewhat uniform across different types of formal logic. The differences in difficulty found were related to the type of statement and were specific to the age of the child. The difficulty increased significantly when a statement was negated.

O'Brien and Shapiro (1968) agreed with many of Hill's findings; however, their results indicated little growth in logical reasoning ability between the ages of six and seven. They found that children have great difficulty in deciding whether a conclusion is logically necessary; moveover, they cautioned that one should not take for granted deductive ability in young children. Their results appeared consistent with Piaget's theory of the development of formal logic in children.

Shapiro and O-Brien (1970) again modified Hill's study on senential logic, syllogism and logic of quantification and in general confirmed findings of the earlier study when similar tasks were employed. On the other hand, when they introduced a new type of response called "not enough clues" they observed a different growth curve. They concluded that children could recognize logical necessity much more easily than they could test for it.

McAloon (1969) introduced units of logic to third and sixth graders. The units included work with sets and set relations, truth and falsity of statements, the meaning of all, some and none, conjunction and disjunction, implications and negations and the fundamental concepts of validity and invalidity. The children who were taught logic attained significantly higher scores on mathematics achievement and on logical reasoning than the children who were not. Mathematical achievement was increased even though some time was taken away from mathematics instruction in order to develop the units on logic. At both grade levels class reasoning was easier than conditional reasoning in general. Symbolism and principles of transitivity were not difficult at either grade level. Fallacy principles and the concepts of converse and contra-positive were more difficult for third-graders than for sixth-graders. The most difficult items were those with a correct answer of "maybe."

McCreary (1965) and Snow and Rabinovitch (1969) studied the difficulty level of conjunctive tasks (perceiving attributes common to positive instances) and disjunctive tasks (perceiving attributes never present in negative instances). In both studies, one with third grade children and one with children of ages five through 13, the researchers reported that disjunctive tasks were significantly more difficult than conjunctive tasks.

In a training study Weeks (1971) concluded that work with attribute blocks strongly affected logical ability and perceptual ability for children in grades two and three.

Gregory and Osborne (1975) investigated the relationship between the frequency of use of the language of conditional logic by seventh grade teachers and their pupils' development of reasoning ability. Student scores on a reasoning test correlated significantly with the frequency of the teacher's use of the language of conditional logic.

Eisenberg and McGinty (1974) compared and contrasted error patterns in logic of prospective elementary teachers and second and third grade children on a test of 30 items in sentential logic. The results indicated that the college students scored significantly higher than the elementary children on two types of items; however, there was little difference in the performance of the two groups on three other types. The researchers concluded that maturation is not a comprehensive factor in the development of logical thinking. They noted that the college students and the children made similar kinds of errors within different types of forms under consideration.

McGinty (1977) reported that second and third grade pupils can learn to answer successfully items from sentential logic when they have been instructed in the use of selected materials. Each of three treatment groups were taught through one of these approaches: (1) the Furth materials, (2) the Dienes materials or (3) a set theory program. The different types of instruction positively affected the performance of children on items from sentential logic, perceptual reasoning and classification; moveover, there were indications of retention over a period of time. The researchers interpreted their study as an indication that young children can profit from instruction in logic. They recommended that such instruction should allow children to explore questions and formulate conclusions through manipulation of concrete materials.

4.22 What has been ascertained about teaching decimals and percent?

The use of decimals instead of common fractions considerably facilitates the computation and comparison of fractional numbers, since the denominator of a decimal is understood to be a power of ten. Decimals are widely used in daily life as well as almost exclusively in scientific and technical work. This useage will increase as the metric system becomes more widely used.

The research on decimals is somewhat limited. Grossnickle (1932) identified common errors that students make in division with decimals. He emphasized the importance of meaningful instruction regardless of which method of decimal point placement was used. The common methods of decimal point placement in division include (1) subtracting the number of decimal places in the divisor from the number of places in the dividend, (2) inserting a caret and (3) multiplying both divisor and dividend by a power of 10 to make the divisor a whole number. Flournoy (1959) reported that students taught to make the divisor a whole number by multiplying by a power of 10 and then multiplying the dividend by the same number resulted in greater accuracy than the subtraction method. For the below average arithmetic achievers, the subtraction method was decidedly more difficult. He noted that the subtraction method seemed to provide more opportunity for error in placing the decimal point in the quotient. A comparison of the decimal-common fraction sequence with conventional sequence of common-decimal fractions at the fifth grade level by Wilson (1969) revealed no significant differences between the two sequences.

Situations involving percents are frequently encountered in daily life, and adults are at a loss if they do not understand the language of percent. A study of percentage by Kennedy and Stockton (1953) with seventh grade pupils revealed that a method emphasizing both drill and understanding was mor effective than either single approach. However, the results were not conclusive. Tredway (1959) reported that instruction involving a treatment of the elements of percent (number, percent and part) was more effective than the conventional textbook approach. McMahon (1959) reported no significant differences between the ratio and conventional methods of presenting percent on tests of interpreting statements about percent; however, the ratio method resulted in greater skill in computation and better retention. Wynn (1966) found no significant differences in achievement or retention between the formula, decimal or unitary analysis methods. McCarty (1965) found that the ratio method was effective in teaching percentage at grades four, five and six.

4.23 What approaches and techniques are most effective in teaching algebra?

A variety of studies have investigated the relative effectiveness of selected strategies for teaching algebra. Meconi (1967) found that high ability students learned and retained concepts related to problem solving performance regardless of whether they were taught by rule-example, guided discovery, or rule-only approaches. Patterson (1969) found no significant differences between discovery and expository groups. The discovery groups used a program on the field axioms with clues given when students needed them, while the expository groups only answered a set of questions. Neuhouser (1965) reported that students taught by the discovery approach, with no verbalization of rules, scored significantly higher on understanding, transfer and retention than students taught by the expository approach and by the discovery approach, where rule statements were required. Lackner (1968) found that the example-to-rule approach resulted in higher achievement than the rule-to-example approach with students in grades 11 and 12. Ballew (1966) found that using discovery exercises in algebra classes over an 18-week period resulted in significant improvement in critical thinking abilities. However, this improvement was not evidenced in similar classes taught using the expository approach. There was no significant difference in the two groups in their mathematics achievement.

In a study of the effectiveness of teaching verbal problem solving in ninth grade algebra, Ashton (1962) found classes using the heuristic method improved significantly more on a verbal problem solving test than classes using the textbook method. The heuristic approach involves a series of questions designed to lead students to discovery by means of plausible or inductive thinking. The students were asked to answer such questions as "What are the data?", "What are the conditions?" and "What is the unknown?" The textbook approach involved a demonstration of methods for solving problems in the textbook with assignments of similar problems from the textbook.

Other studies have attempted to investigate the best method for teaching certain topics in algebra. For example, Nelson (1969) reported that a visual approach to the concept of function was more effective than verbal, numerical or eclectic approaches. Reeves (1969) found that secondary textbooks generally develop functions in a very abstract form as sets of ordered pairs.

Researchers have also attempted to assess the effects of acceleration and enrichment on the study of algebra. Friesen (1961) reported that high ability eighth graders achieved as well as, or better than, ninth grade students in algebra. Klausmier and Wiersma (1964) found that talented students who took the equivalent of three years of work in two yers achieved as well as students in the regular program in algebra, but less well in geometry. Ray (1961) reported that both enrichment and acceleration could prove beneficial to eighth and ninth graders.

Although in the past 40 years less attention has been devoted to developing and maintaining manipulative skills and techniques in teaching algebra, Leonard (1967) found that a group of students in 1966 performed significantly better than did a compariable group who took the same test involving skills in solving problems 40 years before.

4.24 What approaches are most effective in teaching geometry in the secondary school?

The traditional secondary geometry course has become a topic of controversy for both mathematicians and educators. Although this course has been of a Euclidean synthetic nature for many years, currently, what to teach, how to teach it and teacher preparation have become real problems. The thirty-sixth yearbook of the National Council of Teachers of Mathematics, *Geometry in the Mathematics Curriculum* (1973) was devoted to this controversy. In fact, many articles have been written, speeches given and proposals made concerning what should or should not be done in this course of mathematics.

The alternatives to the conventional synthetic Euclidean geometry course include approaches using coordinates, transformations, vectors and combinations of these approaches. The disparities in points of view relate to both content and methodology. The study of geometry in the secondary school has been a controversial issue in the past 50 years, and it appears the debate will continue.

The emphasis on how students learn mathematics has focused attention on the role of informal geometry in the secondary school. Peterson (1973) traced the historical development of informal geometry in grades seven through nine and noted the continuing role of informal geometry in grades 10 through 14. Trafton and LeBlanc (1973, p. 13) noted that informal geometry can be justified from three different points of view. First, the investigation of geometric facts and relationships in an intuitive and exploratory manner provided a good foundation for a formal study of geometry. Second, the learning of geometry should emphasize relationship between geometry and the world of the student, and an informal approach involving concrete materials and models was more likely to accomplish this. Third, informal geometry lends itself to illustrating many practical applications of geometry and other mathematical ideas used in science, technology and work. Cheatham (1970) reported no significant differences between constructing models with compass and straight edge than with paper-folding techniques.

The research on geometry has increased since the mathematics curriculum reform which placed more emphasis on geometry, particularly at the elementary and junior high levels. Davis (1969) investigated the ability of children in grades six, eight and 10 to visualize plane sections of selected figures. He found that sixth graders scored significantly lower than students in grades eight and 10. In a similar study, Palow (1969) reported that the ability to visualize planar sections of solid figures developed at about age 12. However, Boe (1966, 1968) found that students in grades eight, 10 and 12 could not draw and identify geometric sections with consistency.

To assess the changes that have taken place in geometry instruction, Neatrour (1968) surveyed 16 textbooks series and 156 middle schools to obtain information. He found that about three times as much geometry was taught in 1968 as in 1900. A report by Quast (1968) provided an analysis of geometry in high school curriculum. He found that the geometry curriculum had received criticism and re-evaluation continually since 1890, with very little actual change taking place.

Ahmad (1970) compared the changes in the foundations and fundamental concepts of plane geometry that have occurred since 1930. He reported that textbook treatments have become increasingly rigorous.

Usiskin (1969, 1972) reported that students using regular geometry texts scored significantly higher than students using transformational-oriented texts on standardized geometry tests. There was no difference in perceptual skills between the two groups. There was no significant difference in algebric skills, but possible tendencies favored the transformational groups. The transformational-oriented course content seemed to be no more difficult than the standard content. Kort (1971) followed up this study by testing for retention and transfer in 11 grade mathematics classes. His findings indicated that student attitudes and retention of standard geometry content was not significantly different between the transformational oriented groups and the traditional groups. However, retention on the concepts of congruence and similarity was higher for the transformational groups. The transformational groups also achieved higher on a relations-functions test demonstrating the transfer of transformational ideas.

Williams (1966) found that vectors could be used effectively as unifying agents when taught with a linear algebra emphasis. Bundrick (1968) reported that students using a vector approach in algebra II obtained significantly higher scores on criterion and transfer tests than students using a conventional approach.

Friedman (1976) found that geometry teachers ask relatively few memory questions and tend to ask questions making greater intellectual demands on their students than teachers in other subjects.

Although the research evidence on teaching geometry is somewhat limited, this area still remains one of the most controversial issues in secondary school mathematics. A careful exposition of the major issues surrounding the teaching of geometry can be found in *Geometry in the Mathematics Curriculum* (1973).

4.25 What is the role of applications in school programs?

A major recommendation of the National Advisory Committee on Mathematical Education (NACOME) in their 1975 report entitled *Overview and Analysis of School Mathematics Grades K - 12,* focuses on applications. NACOME, after identifying the classic interdependence that has existed between science and mathematics, points out that newer application areas for mathematics have emerged in such areas as mathematical biology, psychology, sociology and management science. Moreover, the first generation experimental texts of the post-Sputnik era neglected applications. These books, as with most texts, tests and state and local syllabi, identified "applications" with "word problems." According to Henry Pollak and other applied mathematicians, such word problems are often misleading with regard to real applications. In fact, surprisingly few attempts have been made which involve students in real world problem-solving experiences. Examples of such attemtps include the following projects.

1. The Minnesota Mathematics and Science Teaching Project (MINNEMAST)
2. The Unified Science and Mathematics for Elementary Schools (USMES) Program

3. Project ONE

4. The Man-Made World (TMMW) Program

Following a review of these pioneering efforts NACOME (1975) recommended

> ". . . that the opportunity be provided for students to apply mathematics in as wide a realm as possible—in the social and natural sciences, in consumer and career related areas, as well as real life problems that can be subjected to mathematical analysis" (NACOME, 1975, 138).

Because of the potential role of computers in simulating real world problems, another recommendation is germaine, namely, "that **all** students, not only able students, be afforded the opportunity to participate in computer science courses" (NACOME, 1975, 139).

A third NACOME recommendation is also aimed at extending the possibility of involving students in applications. That recommendation is "that instructional units dealing with statistical ideas be fitted throughout the elementary and secondary school curriculum." (NACOME, 1975, p. 139).

The above recommendations carry with them certain questions which, according to the NACOME report, demand investigation. For example, does an applications-oriented curriculum improve interest or attitude? Does applying mathematics enhance understanding of mathematical ideas and/or problem solving ability? Which mathematical skills and concepts appear to be most effectively taught through specific, real life models? Do interdisciplinary approaches rely too much on incidental learning of mathematics?

Clearly, a major attempt to design, implement and evaluate mathematical modeling programs must be undertaken in order to obtain even a first approximation to answering the above questions.

4.26 Should probability and statistics be included in mathematics programs?

Among recommendations for school mathematics programs there is almost universal agreement that probability and statistics be included as a major strand in both elementary and secondary programs. NACOME (1975) states that statistical ideas be integrated throughout curricula at all levels. In particular, one can 1) use statistical topics to illustrate and motivate mathematics learning and 2) emphasize statistics as an interdisciplinary subject through the insertion of statistical ideas into the study of the natural, physical and social sciences and the humanities. In particular, NACOME (1975) suggests providing a ninth grade statistics course to all students—with no algebra prerequisite. NACOME states, "This could probably be the most useful mathematics course for the non-college bound or any student who as consumer and citizen must cope throughout life with numerical information." (NACOME, 1975: 145).

Also NACOME (1975) recommends that a school provide

- a year-long statistics course with a probability prerequisite and/or
- Interdisciplinary courses oriented towards computers and statistics. A major difficulty in implementing such statistics oriented courses is that there is a lack of clear, interesting and elementary written descriptions of examples of statistical activities. A major attempt to remedy this lack has been provided by the Joint Committee on the Curriculum in Statics and Probability of the American Statistical Association and the National Council of Teachers of Mathematics. These two groups jointly produced a series of four pamphlets

entitled, *Statistics by Example* (1973). The subtitles in the series are 1) Exploring Data, 2) Weighing Chance, 3) Detecting Patterns and 4) Finding Models. The first booklet only requires that students be familiar with arithmetic, rates and percentages, whereas the second pamphlet presumes that students be familiar with notions of probability and elementary algebra. The last two pamphlets assume that students have studied elementary probability and intermediate algebra. In all cases, these pamphlets do treat real-life problems seldom found in mathematical texts.

For youths at the elementary level, Lennard Rade and Arthur Engel advocate that all mathematics be pervaded by probability beginning as soon as children become acquainted with fractions. (Rade, 1970: 29). They recommend that the best approach to probabilities is not by way of theorems and formulas. Instead, acting out in the classroom, initially with roulette-wheels and dice, then stylizing through the use of tables of random numbers should be used. (Such an approach is recommended for Georgia students, as will be pointed out in a later discussion of the Georgia Mathematics Guide in the Evaluation section).

With regard to older students, a study concerning the teaching of elementary statistics at the college level was conducted by Joe Dan Austin in 1972 at Purdue University. The study dealt with three instructional methods in which the three treatment groups studied a unit on probability. One group used a manipulative and pictorial approach, another used only a pictorial approach, and a third used a symbolic approach. Results indicated that the manipulative and pictorial approach was superior to the other approaches Austin considered in teaching elementary statistics at the college level. Although at present there is disagreement as to how probability and statistics should be taught, one can safely predict that increasing attention will be addressed to this problem. For as Pollack (1970) points out, if one argues usefulness in the everyday lives of the maximum number of high school graduates, then probability and statistics would be the most important topic in the school mathematics program. Probabilistic and statistical reasoning and ideas are crucial for the large number of judgments encountered in everyday life. (Pollack, 1970: 325)

4.27 What do research studies indicate concerning computer usage and mathematics learning?

Many would argue that there exists a growing need for including "computer literacy" among the objectives of school programs. Hatfield (1973) stated,

> *"The computer is a major force showing the accelerated changes of our society. This leads to the contention that educated citizens of the 'computer generation' should have an awareness of this modern tool." (Hatfield, 1973:1)*

Recently, Hatfield (1970) conducted a study to compare programming effects on seventh graders in their mathematics class. The basic language was used on a time-sharing system. This was a two-year study conducted at the University of Minnesota High School and subjects were randomly assigned to treatment groups. The experimental group wrote computer programs using the same mathematical concepts as the other group which did not use the computer.

During the first year there was no significant difference on scores between the two groups except on Numeration Systems, where the non-computer group scored higher. Hatfield reports that, "Learning basic programming seemed to interfere with the concurrent study of Numeration Systems" (Hatfield, 1970: 4330).

During the second year, the computer treatment group scored higher on Elementary Number Theory, Contemporary Math Test and Thought Problems Test. He summarized his findings as follows.

> *"These results lend support to computer-assisted problem solving in grades seven, particularly for the study of number theory, ability to solve unfamiliar word problems, for performance on problem items and for these high and averae achievers. But special concern for teaching the content in which programming is introduced and for development and maintenance of math skills should be noted." (Hatfield, 1970: 4330)*

Hatfield and Kieren (1972) had conducted a similar study in 1965-67 but with two grade levels—grade seven and grade 11. It was a two-year study, but each year was a different experiment. Similar results were reported here in grade seven as Hatfield confirmed in 1970. The results in grade 11 favored the computer treatment group on 13 of 15 measures, except on the Trig unit in the second year. As far as achievement level and programming is concerned, they stated the following.

> *"Grade seven report indicates that even low achievers can learn to successfully program mathematical problems. Yet the average and above average seventh grade achievers seemed to benefit relatively more from the computer treatment. But the Grade 11 study suggested a positive effect for its average achievers." (Hatfield, 1972: 110)*

Wilkinson (1972) conducted a study to investigate whether or not instruction to programming logic or flowcharting would increase mathematical reasoning ability. The subjects were tenth grade students taking a regular geometry course. Sixty-two students were randomly assigned to the experimental group and 85 students to the control group. Experimental data was obtained by pretest and posttest. One of the results was "evidence of a significant increase in mathematical reasoning ability as a result of instruction in logical flowcharting. (Wilkinson, 1972: 4204-A)

A study by Milner (1972) investigated "the effects of teaching computer programming on performance in mathematics" (Milner, 1972: 4183). Eighteen fifth grade students were involved for 15 weeks using the LOGO language. The experiment consisted of three phases. First, the students were taught the LOGO language; second, they wrote programs and third, they were given problems to solve. Results indicated a significant difference favoring the LOGO group. Milner reported these findings.

> *"Computer programming is an effective learning resource in terms of both cognitive and affective considerations.*
>
> *The particular method used in teaching computer programming is less important than the definition of suitable tasks and the preparation for them.*
>
> *The learner-control inherent in this study facilitated the acquisition of problem-solving behaviors.*
>
> *Some of the students whose motivation was questionable in the traditional classroom were turned on by computer programming." (Milner, 1972: 4184)*

Foster (1972) investigated "problem solving performance of students with regard to non-routine problem experiences in which the computer and flowcharts are used by students as aids" (Foster, 1972: 4239). Sixty-eight students were divided into four treatment groups of non-use of computer or flowcharts, use of flowcharts only, use of computer only and use of computer and flowcharts. He measured nine behaviors associated with problem solving and reported that the group which used the computer only had a higher mean performance than the other three groups, and this mean performance was significantly greater than the group which used neither computer nor flowcharting. Each of the three groups using either computer and/or flowcharting had a greater mean performance than the one that used neither.

Student attitude and achievement were measured in a study conducted by Jones (1972), using 258 high school physics and chemistry students randomly assigned to control and experimental groups. The control group performed 10 experiments using the conventional laboratory tools and the experimental group used computer simulation in the 10 experiments. The following results were stated.

- There was no difference in attitudes in either group toward the subject and toward laboratory work before and after the experiment.
- At start of experiment, there was no difference in attitudes by groups toward the computer as a laboratory tool.
- At close of experiment, the experimental group's attitude toward the computer as a laboratory tool was significantly more positive than the control group's.
- There was no significant difference between mean scores on achievement tests.
- The posttest measured more positive attitude by males than females toward using computers. (Jones, 1972: 4200)

Perles (1975) studied the effects of computer use in undergraduate statistics courses. She stated the following in her summary of conclusions: "Computer usage did not hasten the learning process or make possible the teaching of more instructional topics . . . Teaching computer usage was an addition to the students' education. Computer usage introduced the students to a valuable statistical tool . . . Computer usage greatly improved the quality of instruction in the opinion of the faculty" (Perles, 1975: 7086). She also reported positive effects in student attendance, students' class participation, student attitude and motivation and student use of computer skills in other courses.

Significant differences were not found, but results favored the computer group in a study conducted by Statz (1973). Children of age nine to 11 were taught programming, using LOGO and were tested before and after the experiment in problem-solving tasks. A control group also received the tasks. One hypothesis tested was that the LOGO children would perform significantly better on the problem-solving tasks, and they did so, but in only two of four tasks. A secon hypotheses that the LOGO children would do better on problems having constraints showed a supported tendency.

Byers (1973) compared three approaches of instruction. In the first approach, students learned to write programs. In the second, students used pre-written programs and in the third approach, students were taught by traditional methods and did not use the computer. He stated these results.

> *"Students with prior computer experience received higher scores on the content-oriented exam. Students receiving an extensive training in computers felt that computers made it possible to use more real-world-like homework exercises, that computers were more helpful in aiding the understanding of the material, and that computers were more useful in performing numerical computations. The most effective method appeared to occur in the extensive group." (Byers, 1973: 6938)*

Less encouraging results are reported by Mandelbaum (1973). He studied "the effects, on achievement and attitude, of the use of the computer as a problem solving tool with low performing tenth grade students" (Mandelbaum, 1973: 3700). The control group received regular instruction with no computer use. The experimental group used the computer as a tool to solve problems in their regular course of study. The experiment lasted 15 weeks. He states the following findings.

> *"There was no significant difference in achievement in computation, concepts, or applications, no significant difference in attitudes toward mathematics as a process, in attitudes about the difficulties of learning mathematics or attitudes toward the place of mathematics in society; and no significant difference in rate of attendance." (Mandelbaum, 1973: 3700)*

A study of overall developments and trends concerning the use of the computer was conducted by Moran (1974). He found, "In terms of student-related developments, the picture is less clear. Existing research, though highly positive in tone, does not yet enable the reader to draw generalizations. The use of CEI (computer extended instruction), however, seems to hold potential in enhancing mathematics achievement, problem-solving ability, motivation and creativity." (Moran, 1974: 2104)

5. PROBLEM SOLVING

5.1 What is problem solving?

Research on problem solving appears to rest on the fundamental assumption that most mathematical activity is problem solving; indeed, the effective teaching of mathematics is associated with having students develop and utilize problem solving techniques.

Based on seminal ideas of Karl Duncker (1945), John F. Lucas (1972) stated that, in general, a problem situation exists when one possesses certain given information and a goal but lacks a connection between the two. A solution of the problem results when an individual establishes a meaningful connection between information and goal. The term "problem solving" implies more than seeking a solution. In fact, the process of solving a problem involves an active search for a suitable method of dealing with the problem—and subsequent application of that method. Prior experience may provide appropriate methods, procedures and plans. However, for many mathematical problems an individual must construct and test a variety of plans before one is found to be adequate. Consideration of such plans leads many researchers to the notion of heuristics—which is next examined.

5.2 What skills and abilities are related to problem solving?

Kilpatrick (1969) reported that the relationship between problem solving and its underlying dependent skills—especially higher-order verbal skills—is both complex and little understood. Moreover, if one accepts that problem solving ability depends on reading and computational skills, it is not clear which specific contributions these skills provide for solving word problems. For fourth and eighth graders, Martin (1963) stated that reading comprehension, computation, abstract verbal reasoning and arithmetic concepts were all factors associated with problem solving in arithmetic as assessed by the *Iowa Tests of Basic Skills*. Balow (1964) also found that reading and computational skills were correlated with sixth graders' problem solving abilities as measured by the Arithmetic Reasoning subtest of the *Stanford Achievement Test*.

Weredelin (1969) synthesized two factor analytic studies dealing with high school boys' abilities to do mathematical problem solving. For each study Weredelin isolated five factors with almost identical loadings. A General Reasoning factor was found to be most strongly related to problem solving ability. Deductive Reasoning and a Numerical factor were partially related to problem solving tests, whereas Space and Verbal Comprehension factors were unrelated. Very (1967) found in working with college students that one could more easily differentiate mathematical abilities of males from females. Dye and Very (1968) reported finding a similar pattern for ninth and eleventh graders.

A recent report by Suydam and Weaver (1977) concluded that sex difference and socioeconomic status do not appear to be significant factors in determining problem solving skills of elementary school students. Further, they state that I.Q. is significantly related to problem solving ability. Additional factors—besides I.Q., computational skill and reading comprehension—which may characterize good problem solvers are

1. ability to estimate and analyze
2. ability to visualize and interpret quantitative facts and relationships
3. ability to understand mathematical terms and concepts
4. ability to note likenesses, differences and analogies
5. ability to select correct procedures and data

6. ability to note irrelevant detail
7. ability to generalize on the basis of a few examples
8. ability to switch methods readily
9. higher scores for self-esteem and lower scores for text anxiety (Suydam and Weaver, 1977, 42)

5.3 What does research say about teaching children to solve word problems?

Some different approaches have been taken in the investigation of questions related to solving word problems or story problems. Some representative studies are cited here.

Steffe (1970) studied the relation between first grade children's ability to solve addition problems and their ability to make quantitative comparisons. His results showed that children who failed the quantitative comparisons tests scored significantly lower on addition problem tasks. Similar results had been reported for quantitative comparisons and subtraction problems by LaBlanc (1968).

There is some disagreement as to whether children are able to solve more easily problems in which an action is described in the statement (see Steffe [1970] and Steffe and Johnson [1971]). These studies and others (e.g. Bolduc [1970]) tend to agree that young children are able to solve problems more successfully if they are allowed to use manipulative objects.

Pace (1961) investigated the effect of understanding addition, subtraction, multiplication and division on the problem-solving ability of fourth graders. For the experimental group, periods of systematic instruction were provided during which children read the problems, decided how they should be solved and why the chosen process was appropriate. The children in the control group solved the same problems as those in the experimental group; however, there was no discussion of the hows and whys of solution methods. The results of the study indicated that both groups improved in their ability to solve conventional word problems but that the experimental group showed a greater increase than the control group. The results of the study suggested that children will improve in their ability to solve word problems if they are simply exposed to many problem-solving tasks; however, they may improve even more if the teacher provides systematic instruction aimed at a greater understanding of the arithmetic processes.

Wilson (1964) designed two programs for fourth grade children. In each of the programs, which included one-step addition and subtraction problems, there was an attempt to create a mental set. In the first program the learner was taught to perceive the "action-sequence structure" of the problem, express this perception in an equation and perform the operations indicated by the equation. In the second program the learner was taught to recognize the "wanted—given" relationship inherent in the problem, express this relationship in an equation, and perform the operation indicated by the equation. The results of the study indicated that the "wanted—given" treatment was superior for all types of problems represented on all variables measures—choice of operation, correct answers and speed. The results were consistent at high, medium and low mental age levels.

Carpenter et al (1975; 1976) discussed results of the first National Assessment of Educational Progress (NAEP) relative to children's success in solving word problems requiring whole number computations. The exercises were given orally on a tape recorder so that reading ability would not cause major difficulty. The researchers made the following observations concerning the responses of nine-year-old subjects and 13-year-old subjects in the study.

> The nine-year olds tended to perform at a lower level than the 13-year-olds, were more likely to respond "I don't know," and made more errors in selecting the operation, doing the computations and recalling number facts. In both age groups, the performance on word problem exercises tended to be slightly lower than the corresponding performance on computation exercises requiring the same operations (1975, p. 444).

The researchers stated a clear bias in favor of emphasizing problem-solving skills.

> Clearly, the student deserves the right kind of problem-solving experience and the teacher has the central responsibility in constructing that experience—finding interesting problems, asking probing questions, getting students to relate to and enjoy working with word problems . . . (1975, p. 392).

They recommended the following practices to every teacher.

1. Do not deny children an opportunity to solve word problems because of lack of reading skills. Read the problem to nonreaders. Help poor readers use problems in improving their problem-solving ability as they improve their reading skills.

2. Use word problems in such a way as to develop or reinforce understanding and skill in computation.

3. Have children check their work and discuss the answers and procedures for obtaining the answers.

4. Encourage children to explore alternative solutions, to alter the data and to make up other problems that are most interesting to them.

After examining numerous studies of different types of problem solving in the elementary school, Suydam and Weaver (1975; 1977) offer the following practical suggestions, although they acknowledge that there is little hard research data to support all of the recommendations.

1. Provide a variety of problems at appropriate difficulty levels.
2. Have students write the mathematical statement or number question for each problem.
3. Allow students to dramatize problems and their solutions.
4. Encourage children to make drawings and diagrams of the problem situations and to use them in verifying their solutions.
5. Give conditions and have pupils formulate appropriate problems.
6. Provide problems without numbers.
7. Have students indicate the process that is to be applied.
8. Ask students to test how reasonable their solution is.
9. Allow pupils to cooperate with each other in solving problems.
10. Suggest that students find alternate ways to solve problems.
11. Give children opportunities for analyzing situations that give rise to real problems rather than merely providing computational practice.

12. Give oral problems.

13. Ask children to tell what essential information is missing from incomplete problems and what information is unnecessary when more is given than is needed.

14. Present problems on a tape recording to help poor readers.

They also list the following common reasons why children make mistakes (1977, p. 42).

1. Errors in reasoning

2. Ignorance of mathematical principles, rules or processes

3. Insufficient mastery of computational skills

4. Inadequate understanding of vocabulary

5. Failure to read for noting details

Suydam and Weaver (1977) also noted that hand-held calculators may plan an important part in problem solving activities in the elementary school. As calculators become increasingly available in classrooms, it will be possible for teachers to emphasize problem solving in a broader sense by refocusing attention from word problems that simply provide applications of computational skills to real world problems that involve structuring and analyzing situations as well as finding a solution.

5.4 What does research on problem solving imply for students in school programs?

The area of problem solving is perhaps the most extensively researched topic in mathematics education. Although the quality of this research falls far short of the quantity produced, it appears that certain results are reportable.

Suppes, Loftus and Jerman (1969) found that number operations and other structural variables contributed less in determining a particular problem's difficulty level than did the similarity of a previous problem to the given problem. Steffe (1967) reported that first graders could more easily deal with oral problems when a common name was used for sets and when physical or pictorial aids were provided.

Suydam and Weaver (1975, 1977) and Suydam (1974) have ambitiously attempted to assess the vast literature on problem solving as it relates to elementary programs. Among their many findings are that children (1) enjoy a variety of problem settings, (2) do better when data in a multi-step problem is presented in the order required for use, (3) take less time when a question is posed at the beginning of a problem rather than at the end, (4) achieve independently of the positioning of a question in a problem, (5) become more adept at problem solving if encouraged to structure, to analyze and to solve problems in a variety of ways and (6) are often misled by isolated word cues (such as **left** or **in all**) and consequently fail to notice crucial relationships.

5.5 What is heuristic problem solving?

While most researchers agree that the search for a plan is the crux of the solution process, there are basic rules-of-thumb, strategies and techniques which serve to guide and focus the search by reducing the field of alternative approaches and procedures. This reduction process, which involves selecting and ordering alternatives, may be referred to as heuristic problem solving.

Concern for such problem solving in mathematics has been most influenced by the numerous writings of George Polya. For example, Polya states that problem solving constitutes "finding a way out of difficulty, a way around an obstacle, attaining a aim which was not immediately attainable" (Polya, 1962, v).

In his classic book *How to Solve It* Polya remarked that the word "heuristic" has been used as an adjective that means "serving to discover" (Polya, 1957, 113). Both Polya and Jon Higgins (1971) point out that as a noun form heuristic or "ars inveniendi" was the classic name of a branch of study involving logic, philosophy and psychology.

> *"Modern heuristic endeavors to understand the process of solving problems especially the* **mental operations typically useful** *in this process. It has various sources of information none of which should be neglected . . . but it should least neglect unbiased experience. Experience in solving problems and experience in watching other people solving problems must be the basis on which heuristic is built." (Polya, 1957, 129-130)*

Kilpatrick (1967), after elaborating the difficulties and subtleties of determining what is meant by heuristic, proposed the following definition: "A heuristic is any device, technique, rule of thumb, etc., that improves problem solving performance" (Kilpatrick, 1967, 19).

A main concern of studies dealing with heuristic (e.g., Lucas, 1972 and Kantowski, 1975) is that heuristic processes are tentative; that is, the processes do not necessarily lead to or guarantee a solution, but they do enhance the discovery of an appropriate **method** of solution. That is, if one defines an algorithm as a process that leads to a solution after a finite sequence of steps, then algorithms are non-tentative processes. By contrast, the heuristic of "draw a figure" may or may not aid the problem solver and thus is a tentative procedure.

Because of the growing awareness of the importance of heuristic problem solving and because so few current programs concern themselves with this approach—stressing instead an algorithmic approach—one may assume that future research and development efforts will respond to this concern.

5.6 What research has focused on heuristic problem solving?

Many investigators (for example, Rosenbloom, 1966; Richards, 1968; Bernard, 1972) have indicated that teaching heuristic problem solving should be a main goal of the total education effort. All of these writers have been influenced by the works and spirit of George Polya (1957, 1962, 1965) in stressing and promoting heuristic processes. Investigators such as Ashton (1962), Libeskind (1971), Lucas (1972) and Goldberg (1974) all found that problem solving performance was improved through use of an instructional procedure based on Polya's heuristic process. For example, Ashton reported that ninth grade algebra students who received 10 weeks of instruction stressing heuristic processes evidenced greater ability to solve word problems when compared to a traditionally taught group.

According to Kilpatrick (1969),

> *"The most impressive evidence for the validity of Polya's observations on the problem-solving process has come from work on computer simulation of human behavior. Programmers have found that the incorporation of general heuristic rules, such as working backward or using a diagram, not only facilitates problem solving, but also results in performance by the computer that closely resembles the behavior of humans struggling with similar problems." (Kilpatrick, 1969, 527)*

In researching heuristics as applied to non-mathematical problems, Covington and Crutchfield (1965) used the Productive Thinking Program (PTP), programmed booklets which employ a comic-book format. The fifth and sixth graders in the treatment group evidenced superiority in originality, divergent thinking and in perceiving the value of problem solving. Kilpatrick (1969) reports that other workers have been unable to replicate Covington and Crutchfield's significant findings. Olton (1969) used a revised version of the PTP and obtained substantial results with regard to Polya's "looking back" heuristic.

Wilson (1967) investigated the question of how specific or how general heuristics should be. Subjects studied two theorem-proving tasks, one on logic and one on elementary algebra, using self-instructional booklets. Subjects were taught, for each task, one of three kinds of heuristics: (1) task specific, (2) means-end (reduction of difference between given and goal), and (3) planning (i.e., general proposed solution which omitted details). Wilson found that a combination of heuristics aided performance on some transfer tasks and that general heuristics learned in the first training task were practiced on the second task, thereby facilitating transfer.

Basing his work on that of Wilson, Smith (1973) was unable to support the hypothesis that differing the levels of heuristic instruction led to differential performance on transfer tasks. Smith concluded that the apparent lack of transfer of general heuristics may be due to our present ignorance concerning human problem-solving behaviors; thus greater attention should be given to little-understood processes than to observable products.

Kilpatrick (1969) analyzed problem-solving protocols of junior high school students with regard to achievement, attitude and aptitude. He found that subjects who employed the most trial and error evidenced greater achievement than those who employed the least trial and error. In fact, this latter group did poorest on word problems. One should note, however, that subsequent studies on the efficacy of trial and error are quite mixed.

Lucas (1972) and Goldberg (1973) used modified versions of Kilpatrick's (1969) coding and analyzing system in exploratory research on college level study of calculus and number theory. Overall, Lucas found that subjects in the heuristic treatment group scored higher on problem solving tasks with respect to approach, plan and result. The treatment group made greater use of certain heuristics such as working backward, comparing with a related problem and using appropriate notation. However, it was not apparent that other heuristics (such as use of diagrams, trial and error and the ability to construct proofs) were related to treatment. Goldberg (1973) randomly assigned subjects to one of three different groups: reinforced heuristic, unreinforced heuristic and non-heuristic. She found that the reinforced heuristic group did relatively better than the other two groups with respect to concepts, writing proofs and attitude toward problem solving.

Examination of recent literature on problem solving in mathematics shows, as Smith (1973) proposed, progressively more concern with behaviors than with products. This trend is illustrated by studies conducted by Vos (1973) and Webb (1975) with secondary school students. Perhaps the most focused and relevant clinical exploratory study was conducted in 1974 by Mary Grace Kantowski and reported in Kantowski (1977). She investigated processes involved in solving complex, non-routine problems by analyzing verbal and written protocols of eight high-ability ninth grade algebra students. Following a clinical methodology much used by Soviet investigators, subjects were asked while solving problems to "think aloud" and their protocols were analyzed later by a modified Kilpatrick (1967) coding scheme. A process-product score was based on (1) suggesting a plan of solution, (2) persistence, (3) looking back, (4) absence of structural errors, (5) absence of executive errors, (6) absence of superfluous deductions and (7) correctness of result.

Kantowski found that goal-oriented heuristics (i.e., specifically related to the conclusions of the problem) led to more efficient solutions. Also, a subject's tendency to use goal-oriented heuristics increased with the development of problem-solving ability. Overall, she states:

> *"The introduction of a heuristic related to the goal seemed to correspond to 'insight.' From that point, the path to the goal was, in most cases, clear and characterized by regular analysis-synthesis patterns" (Kantowski, 1977, 166).*

By analysis she meant what Polya refers to as decomposing, or making inferences from what is found to be at hand, whereas synthesis is the recombining of problem elements to form a new whole. This striking evidence of possible regularity of analysis and synthesis could be related to some significant patterns generally employed by high-ability problem solvers.

Kantowski also observed another regularity. She stated, "Persistence did seem to be affected by prerequisite knowledge and by personality factors" (Kantowski, 1977, 69). In particular, she noted that "reflexive individuals tend to be more persistent on difficult tasks than impulsive subjects" (Kantowski, 1977, 169). The use of the "looking back" heuristic for a more elegant proof or another solution did not increase with problem-solving ability. She conjectured that this strategy (i.e., checking that one is correct) involves a strong affective factor not felt as a need by the novice problem solver.

Kantowski concluded her interesting study by recommending numerous further clinical exploratory studies. For example, she recommended that the regularities she observed be repeated with students of average and weak ability. Krutetskii (1969) maintained that problem-solving aptitude is directly related to mathematical aptitude. However, warned Kantowski, any such future endeavor calls for the development of reliable instruments for assessing processes. Only then, she said, will one be able to determine if the use of heuristics is in fact a common factor in successful problem solving.

6. EVALUATION

6.1 What are definitions and current directions in evaluation?

The primary purpose of evaluation in education is to provide information for decision making. Information should be useful for improvement, continuation and termination decisions according to Anderson, Ball, Murphy, (1973). The evaluation procedures should be objective, reliable, valid, practical, useful and ethical.

Two categories of evaluation are formative evaluation and summative evaluation. Anderson, et al., define formative evaluation as involved with helping develop programs or products through empirical research methodology. Summative evaluation carries out evaluations on an operating program.

Popham (1975) presents a definition of systematic educational evaluation that "consists of a formal assessment of the worth of educational phenomena." His conception of evaluation focuses "determination of merit" of educational procedures.

Regardless of what theoretical definition one employs, as Jackson (1968) pointed out in his investigation of classroom situations, one important aspect of school life is evaluation. In fact, students are evaluated as soon as they begin school, and such evaluations accumulate throughout their school years. "The most obvious difference between the way evaluation occurs in other situations is that tests are given in school more frequently than elsewhere . . . Tests are as indigenous to the school environment as are textbooks or pieces of chalk." (Jackson, 1968: 19)

Recommendations of NACOME (1975) suggested that the following aspects of evaluation should receive more attention.

1. Evaluation instruments should be matched to previously identified goals.
2. Students' grade-level scores on standardized tests should not be reported.
3. Objective directed tests should be developed to replace norm-referenced tests.
4. Program evaluation should make use of sampling techniques to minimize over-testing.
5. Evaluation results should be representative of multiple goals.
6. Potential for cultural bias in testing should be minimized.
7. Effects of testing conditions, e.g., time, motivation environment, over testing, should be considered in the evaluation process.

6.2 How are the state guides for mathematics related to the statewide criterion-referenced tests?

In 1972 the state guide, *Mathematics for Georgia Schools,* Volumes I and II, (Georgia Department of Education 1971) was sent to each school system in Georgia. *Mathematics for Georgia Schools* was designed to provide a framework for school systems as they make decisions about local curriculum. Its two volumes, I for grades K - 4 and II for grades 4 - 8, contain objectives with activities keyed to those objectives to aid the teacher in instruction. With the active involvement of more and more school systems in development of mathematics curriculum came the question of evaluation.

Georgia's Statewide Testing Program had been based on norm-referenced tests in grades four, eight and 11. The reporting for norm-referenced tests is in terms of grade equivalency and percentiles. A grade equivalency score can be reported on a given fourth grade student, fourth grade class and all the fourth grade classes in a school, system or state. Such a score provides information about how well an individual or group has performed with respect to another group on whom the test was normed. This does not, however, provide information about a student's mathematics program, i.e., if the student can add, tell time, identify standard geometric shapes or needs help in any or all of these areas. Additionally, school systems often desire some kind of data to help identify possible gaps in their total mathematics program. Thus, the need for criterion-referenced tests.

Educational Testing Service (ETS), Princeton, New Jersey, was contracted by the Georgia Department of Education to begin working with a committee of Georgia mathematics educators to develop criterion-referenced tests (CRT) in mathematics for use in statewide testing in grades four and eight. The committee included primary, middle and upper grade elementary teachers, secondary mathematics teachers, administrators, mathematics supervisors and mathematics educators. Under the direction of Dr. Ann McAloon of ETS, the committee developed objectives for testing. The basis for these objectives was *Mathematics for Georgia Schools*. Objectives were written for each of the six strands—Sets, Numbers and Numeration; Operations, Their Properties and Number Theory; Relations and Functions; Geometry; Measurement; and Probability and Statistics. A large sample of teachers from throughout the state rated the objectives from most desirable to least desirable. The top 20 objectives (representing the six strands of *Mathematics for Georgia Schools*) were selected for test item writing.

The next section centers on the role of the CRT in school programs, and, in particular, deals with the following objectives: "Reads and writes names of numbers up thorugh one million" and "Finds to the nearest number of units a measurement of time, weight, length, area, volume, temperature or money."

6.3 What is the role of Statewide Criterion-Referenced Tests in school programs?

When one notes the present activity and concern with criterion-referenced tests (CRT) it is perhaps surprising to note that as recently as 1969 Romberg, in discussing research in mathematics education stated,

> *"The area of mastery learning and criterion-referenced tests is just now becoming of interest to many measurement specialists." (Romberg, 1969, 484)*

Examination of the state total CRT mathematics results of the Georgia Statewide Testing Program reports for spring 1976 and spring 1977 reveals a high degree of consistency in both grades four and eight for both years. For example, percentage figures spanning the 20 objectives for fourth grade in spring 1976 range from a low of 43 on Objective three to a high of 90 on Objective 10. Corresponding percentage figures for the same two objectives for spring 1977 are 47 and 92, representing again the lowest and highest figures, respectively. The pattern of a slight increase from 1976 (N = 76 228) to 1977 (N = 80 103) is consistent in that only two objectives increased by five percentage points, whereas eight objectives increased by four points, three objectives increased by three points, six objectives increased by two points, and lastly, one objective increased by one point.

Comparison of the eighth grade results for spring 1976 and spring 1977 are even more striking in that they are significantly close to each other. For spring 1976 (N = 88 703) the lowest percentage score was 28 (Objective 5) and the highest was 84 (Objective 1). Corresponding results for the same two objectives in spring (N = 88 418) were, respectively, 29 and 86. Similarly, all objectives showed either no gain or a slight gain to a maximum of three points on a single objective. In particular, 12 of the 20 objectives gained only a single point, three objectives gained two points, and four objectives were identical in percentage points for both years.

As stated above, an analysis of results for fourth grade showed that Objective 3 was the least attained objective across the state. This particular objective was the following: "Reads and writes names of numbers up through one million." To achieve this objective a student is required to successfully answer four of the six test items associated with this objective. Each school is provided with the number of students of those tested who did attain this objective. Moreover, specific and general diagnostic suggestions are provided to aid in having both individuals and groups of students successful with respect to this objective. For example, for one school which tested 58 fourth graders only 19 percent of the students met Objective 3. The printout for that school recommended that of the 58 students that (1) 47 may need help in studying the decimal numeration system and (2) 45 may need work with physical models for place value.

The next least met objective on the fourth grade CRT was Objective 16, which calls for finding "to the nearest number of units a measurement of time, weight, length, area, volume, temperature or money." To attain this objective requires answering correctly five out of six test items. Specific suggestions for students failing to meet this criterion include increased work (1) using ruler, (2) using thermometer, (3) telling time, (4) covering surfaces with square units, (5) building blocks using cubic units and (6) dealing with money.

Objective 3 discussed above is one of three objectives in the Sets, Numbers and Numeration strand. Objective 16 is in the Measurement strand. Consideration of any program in elementary mathematics should attend to these two strands. Similarly, specific examination of CRT results for individual students, schools or systems should prove fruitful. The remaining four strands deal with (1) Operations, Their Properties and Number Theory, (2) Relations and Functions, (3) Geometry and (4) Probability and Statistics. Cutting across each of these strands are such processes as computing, problem solving and communicating mathematics. Similar remarks concerning the potential use of the CRTs as diagnostic devices apply to the eigth grade CRTs.

Clearly a careful examination of the results of CRT can benefit the individual student as well as provide teachers, curriculum developers and administrators with a potential guide for program evaluation, modification and improvement.

6.4 What are the basic mathematical skill areas?

The National Council of Supervisors of Mathematics (1977) has produced a position paper which, based on the NIE's Euclid Conference Report (1975), identifies the following as the 10 basic skill areas together with excerpts of the rationale for each area.

1. **Problem Solving**
   Learning to solve problems is the principle reason for studying mathematics . . . . Students also should be faced with non-textbook problems . . . be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions, and they must be willing to subject these conclusions to scrutiny.

2. **Applying mathematics to everyday situations**
   The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics and interpret the results in light of the initial situation.

3. **Alertness to the reasonableness of results**
   . . . Students should inspect all results . . . check the reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

4. **Estimation and approximation**
   Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide wen a particular result is precise enough for the purpose at hand.

5. **Appropriate computational skills**
   Students should gain facility with addition, subtraction, multiplication and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator . . . . Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

6. **Geometry**
   Students should learn the geometric concepts (they) will need to function effectively in the three-dimensional world. . . . They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills.

7. **Measurement**
   As a minimum skill, students should be able to measure distance, weight, time, capacity and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

8. **Reading, interpreting and constructing tables, charts and graphs**
   Students should know how to read and draw conclusions from simple tables, maps, charts and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts and graphs.

9. **Using mathematics to predict**
   Students should learn how elementary notions of probability are used to determine the likelihood of future events. . . . They should become familiar with how mathematics is used to help make predictions such as election forecasts.

10. **Computer literacy**
    It is important for all citizens to understand what computers can and cannot do. . . . The increasing use of computers by government, industry and business demands an awareness of computer uses and limitations.

(National Council of Supervisory of Mathematics, 1977, 20)

6.5 What mathematical topics should be included in essential or desirable criterion-referenced test objectives for secondary mathematics programs?

A student should be able to

- use the four arithmetic operations.

- compute sums and differences of numbers written as decimals, mixed numbers, fractions and percents.

- compute the product of numbers written as decimals, mixed numbers, fractions and percents.

- compute quotients of numbers written as decimals, mixed numbers, fractions and percents.
- demonstrate an understanding of decimal, fractional and percent notation; translate numbers from one mode to another (includes ordering and rounding).
- compute simple and compound interest on savings accounts, bonds, stocks and loans, and determine service charges on installment buying using tables, estimation or a calculator.
- computer profit and discounts as a percent of either the cost or selling price; determine the selling price when the cost and the percent are given (includes computing amount of sales tax using tables, estimation or a calculator).
- read, interpret and complete federal and state income tax forms and withholding exemption forms.
- perform the computational aspects of personal finance such as calculation of wages, reconciling a checkbook and completing a bank deposit slip.
- compute the range, mean, median and mode of given or collected data and recognize uses and misuses of these terms in the interpretation of data.
- state the probability of outcomes, given a description of a probability experiment.
- read and interpret line graphs, bar graphs and circle graphs.
- identify situations in which sampling may affect interpretation of data.
- locate points in a Cartesian plane including maps.
- read and interpret flow charts, tree diagrams, factor trees and Venn diagrams.
- select appropriate units of measure to determine length, area, volume, perimeter, circumference, angle, time, mass, temperature and capacity.
- convert measurement of length, area, volume or time within the customary system or within the metric system.
- estimate measurement with a reasonable degree of accuracy.
- identify and classify sets of points including points, lines, planes, three-dimensional figures, line segments, open curves, closed curves, angles, triangles, rectangles, squares and circles.
- compute the perimeter and area of a simple geometric figure when given dimensions of its sides, and vice versa, using either customary or metric units.
- identify, given two related point sets, the relations: inside, outside, parallel, perpendicular, similar and congruent.
- distinguish between explanations based on personal experience and those based on scientific studies.
- identify information required to solve given problems, and use graphical representations and other data in problem solving situations.

- use data and observations to make decisions and predictions and differentiate between a logical and illogical presentation or conclusion.

6.6 What current recommendations should be considered in evaluating programs to determine if they are adequately preparing secondary students for college level courses in mathematics.

A 1977 report entitled *Recommendations for the Preparation of High School Students for College Mathematics Courses,* prepared by a joint committee of the Mathematical Association of America and the National Council of Teachers of Mathematics, includes the following statements specifically designed to provide a benchmark for sound and stimulating mathematical training for the college bound student.

1. Mathematics is a highly structured subject in which various concepts and techniques are highly dependent upon each other.

2. The concepts of arithmetic and algebra are basic to all of mathematics. Further work in mathematics and in all areas in which mathematics is used as a tool requires correct performance of basic arithmetic operations, manipulation of algebraic symbols and understanding of what the manipulations mean.

3. Neither conceptual understanding alone nor technical skill alone will suffice in today's world, let alone in tomorrow's. For further work in mathematics, and in many other areas, from business to psychology, from biology to engineering, the ability to use algebra with skill and understanding is essential. Algebra courses in secondary school should include, in addition to the basic topics, polynomial functions, properties of logarithms, exponential and logarithmic functions and equations, arithmetic and geometric sequences and series, the binomial theorem, infinite geometric series and linear and quadratic inequalities.

4. For most students adequate coverage of the topics in algebra requires at least two years of study.

5. Students who may eventually take calculus—and this now includes very many students who will take college work in business, premedicine, economics, biology, statistics, engineering and physical science—will need a good deal of what is often called precalculus including a sound understanding of the concept of a function.

6. All college bound students should be introduced to some axiomatic system and to deductive reasoning. Traditionally, this has been accomplished in a geometry course. Geometry courses should include, in addition to basic topics, fundamental properties of geometric figures in three-dimensions, applications of formulas for areas and volume and experience in visualizing three-dimensional figures.

7. Those students needing trigonometry should study trigonometric functions and their graphs, degree and radian measure, trigonometric identities and equations and inverse trigonometric functions and their graphs.

8. Other courses beyond algebra, geometry and trigonometry should be available, such as coordinate (or analytic) geometry, probability, statistics, elementary finite mathematics, linear algebra, introduction to computers and computing and applications of mathematics.

9. Calculus, where offered in secondary schools, should be at least a full year course.

10. All courses should emphasize inductive as well as deductive reasoning, techniques of estimation and approximation, problem-solving techniques, transition for the verbal form to the language of mathematics and applying standard techniques and understanding of important concepts.

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APPENDIX

OBJECTIVES, STATEWIDE TESTING—CRITERION-REFERENCED TEST

Fourth Grade
Mathematics

1. The student can demonstrate that the number of objects in one set is the same as or is not the same as the number in another set; count by one's, two's, three's, fives's, ten's and hundred's.

2. The student can express names of numbers including whole numbers, fractions and decimal fractions in various ways.

3. The student can read and write names of numbers up through one million (with numerical symbols and words).

4. The student recognizes which arithmetic operation is appropriate to a given problem situation.

5. The student can recall any of the addition and subtraction facts and any of the multiplication and division facts through products to 50.

6. The student can apply and recognize use of the properties of numbers (such as properties of zero and one) and properties of operations (cummutative, associative and distributive without emphasis on the use of the words).

7. The student can add and subtract with numerals up to four digits (with regrouping).

8. The student can multiply a three-digit number by a one-digit number, and divide a three-digit number by a one-digit number (with or without remainder).

9. The student can state the relation of a given set of elements; state pairs of elements for a given relation (such as "is equal to," "is less than" and "is the brother of.")

10. The student can sort and classify objects by similarities or differences.

11. The student can make diagrams, tables, graphs or other written recorder of relations (ordered pairs).

12. The student recognizes the names of and can identify standard geometric shapes.

13. The student can select from given geometric shapes a shape which matches one that has been turned around, flipped over, moved sideways, stretched or shrunk.

14. The student can state the relation between points or between geometric figures, such as points inside of or outside a closed curve or a line parallel to another line.

OBJECTIVES, STATEWIDE TESTING—CRITERION-REFERENCED TEST

Eighth Grade
Mathematics

1. The student can make generalized statements using the terms "all," "some," or "none" and distinguish between assumption and consequence.

2. Use numerical forms (fractional parts, equivalent fractions, decimal approximations) of rational numbers appropriate to the given situation.

3. Use fractions or ratios appropriate to the given situation.

4. Select the arithmetic operation appropriate to the given situation or problem.

5. Select a problem situation (from given examples) appropriate to a given mathematical operation.

6. Add, subtract, multiply by 3-digit numbers and divide by 2-digit numbers and compute with simple fractions, decimal fractions, integers and percents.

7. Recognize and apply properties of numbers (such as of zero, one, factors, multiples and primes) and properties of operations (such as commutative, associative, distributive, identity and inverse.

8. Solve simple, one-variable open sentences.

9. Supply missing elements of pairs when given a relation and specify the relation of a set of ordered pairs.

10. Use set notation, rules, formulas, mappings, tables and graphs to identify relations.

11. Identify and classify geometrical figures such as point, line, plane, space, polygon, line segment, open curve, closed curve, angle, triangle, rectangle, square, circle, cube and pyramid.

12. Select from a collection of geometric figures those which are alike under the following: rubber sheet geometry (stretching and shrinking), rotation, reflection, translation and uniform stretches and shrinks.

13. Identify the relation of two given sets of points (such relations as inside, outside, parallel, perpendicular, similar and congruent).

14. Solve simple geometric problems by using direct measurements, approximating measurements, using ratios of similar polygons, and using the Pythagorean Theorem.

15. Apply standard measurement formulas such as perimeter and area of rectangle, triangle and circle; volume of a rectangular solid; and time-rate-distance.

16. Determine measurements of length, area, volume, weight, time, temperature and money using real numbers; and specify reasonable error of measurement.

17. Use measurement to solve problems from other fields such as vocational education and the sciences.

18. Construct and interpret different kinds of graphs; demonstrate how sampling may affect interpretation of data.

19. Identify range, mean, median and mode of given data; recognize misuses of these terms in the interpretation of data.

20. Assign or estimate probabilities of chance events.